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Some Effects of Set-Theoretical Assumptions in Measure Theory

R. DANIEL MAULDIN

Department of Mathematics, University of Florida, Gainesville, Florida 32611

INTRODUCTION

This paper consists of some observations concerning the effect that various set-theoretical assumptions have on measure theory and descriptive set theory. The author was led to the considerations here by a problem stated to him by his advisor, the late Professor H. S. Wall of the University of Texas. The problem as stated by him was to find an integral representation of the dual of functions of bounded variation on the unit interval. This problem has attracted the attention of a number of authors and is stated explicitly by Dunford and Schwartz in their treatise [1, p. 374]. Earlier, in [2, 3] this problem was studied by the author with the aid of the continuum hypothesis. We shall continue here in a similar vein but replace the continuum hypothesis by Martin's axiom or the assumption that the continuum is a real-valued measurable cardinal.

We shall consider integral representations of a more general class of spaces: those spaces consisting of countably additive measures of bounded variation whose values lie in a Banach space having the Radon–Nikodym property. The integral representation problem has led to the study of extensions of various vector-valued measures. Some of the pertinent theorems needed are developed in Section 1.

The problem has also led to the study of a rather natural object, the c-algebra generated by the open subsets of the unit interval. In case the continuum hypothesis holds, this algebra is the classical family of Borel sets. But in case the continuum hypothesis does not hold, the c-algebra is much larger than the Borel algebra and seems to be a natural object of study, particularly if one assumes in addition, the c-additivity of Lebesgue measure or of a c-additive extension of Lebesgue measure to this algebra. Some theorems concerning the generation of this algebra are given in Section 4.

In Section 5, we show that Martin's axiom together with the negation of the continuum hypothesis implies the existence of a lifting of $L_\infty(\lambda)$ into $B(\Sigma_c)$, the Banach space of all Σ_c-measurable functions provided with the supremum norm. It is also shown that the Banach spaces $B(\Sigma)$ and $B(\Sigma_c)$ are not isometrically isomorphic. Finally, a characterization of bimeasurable mappings using Martin's axiom is noted.

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In connection with the integral representation problem the author realizes that there are other approaches which employ only ZFC. The recent results of MacNerney [4] do not involve the cardinality considerations which appear here. The reader is referred to [4] and the references given there and to the discussion given by Dunford and Schwartz [1].

The author would like to thank D. R. Lewis for a number of interesting conversations concerning the results given here. In particular, the use of the conditional expectation operator in Theorem 1.1 was suggested by him. It considerably simplifies an earlier argument of the author. Also, the author wishes to thank K. Kunen for providing Example 3.9.

NOTATION

We shall use the following notations.

τ, κ: infinite cardinal numbers: cardinals are regarded as initial ordinals.

c: the cardinality of the continuum.

X: a set.

I: the unit interval.

S: an uncountable standard topological space: S is a Hausdorff topological space such that there is a continuous injection of a Polish space onto S.

Σ: the σ-algebra of Borel subsets of S.

Σ_κ: the κ-algebra generated by Σ: Σ_κ is the smallest family $\mathscr{F}$ containing Σ which is closed under complements and under unions of less than κ sets from $\mathscr{F}$ (thus, $\Sigma_{\omega_1} = \Sigma$).

E: a Banach space.

E^*: the conjugate space of E.

$\kappa a(S, \Sigma_\tau, E)$: the space of all κ-additive functions μ from Σ_τ into E: if $\{A_\gamma\}_{\gamma\in\Gamma}$ are disjoint sets in Σ_τ, $|\Gamma| < \kappa$ and $A = \bigcup A_\alpha \in \Sigma_\tau$, then $\mu(A) = \sum_{\gamma\in\Gamma} \mu(A_\alpha)$.

$\kappa a(S, \Sigma_\tau)$: $\kappa a(S, \Sigma_\tau, R)$, where R is the reals.

$bv\kappa a(S, \Sigma_\tau, E)$: the space of all κ-additive E-valued measures μ on Σ_τ which are of bounded variation: there is a number M such that $\Sigma \| \mu(E_i)\| \leqslant M$, for every Σ_τ-measurable partition of S. This space will be regarded as a Banach space under the variation norm (which it is).

$ca(I, \Sigma)$: is in particular the space of all c-additive real-valued measures Σ. This is in contrast with the usual notation of analysts.

1. Extension Theorems

In this section we derive some theorems concerning extensions of vector-valued measures which will be needed in the sequel. By a vector-valued measure, we mean a countably additive function from a σ-algebra or ω_1-algebra, $\mathcal{A}$, of subsets of a set X into a Banach space E. Let us recall that if m is a vector-valued measure from $\mathcal{A}$ into E, then

(1) there is a nonnegative countably additive measure μ on such that $\lim_{\mu(A)\to 0} \| m(A)\| = 0$ (m is absolutely continuous with respect to μ);

(2) the range of m is a conditionally weakly compact subset of E.

These results may be found in [5].

Theorem 1.1. *Let Σ be a sub-ω_1-algebra of the ω_1-algebra Δ of subsets of X. Let μ be a nonnegative countably additive measure on $\Delta(\mu \in \omega_1 a^+(X, \Delta))$ and let $m \in w_1 a(X, \Sigma, E)$. Then:*

(1) if m is absolutely continuous with respect to $\mu \mid \Sigma$, then m has an extension $\tilde{m}$ to Δ such that $\tilde{m}$ is absolutely continuous with respect to μ;

(2) if each measure $\nu \in \omega_1 a(X, \Sigma)$ which is absolutely continuous with respect to $\mu \mid \Sigma$ has a unique extension $\tilde{\nu}$ to Δ which is absolutely continuous with respect to μ, then the extension $\tilde{m}$ of (1) is unique.

Proof. Define the operator U from $L_\infty(\mu \mid \Sigma)$ into E^{**} by

$$\langle U([f]), x^*\rangle = \int_X f d\langle m, x^*\rangle.$$

Now, U is continuous and linear. Also since the range of m is a conditionally weakly compact subset of E and $U([\chi_A]) = m(A)$, for each A in Σ, it follows that U actually maps $L_\infty(\mu \mid \Sigma)$ into E and U is a weakly compact operator.

Let α be the conditional expectation operator of $L_1(\mu)$ into $L_1(\mu \mid \Sigma)$. Define $\tilde{m}(A) = U\alpha^*([\chi_A])$, for each A in Δ. Clearly, $\tilde{m}$ is a finitely additive function from Δ into E and $\tilde{m}$ extends m.

To see that $\tilde{m}$ is countably additive, it is enough to show that $\tilde{m}$ is weakly countably additive. Notice that U is a weak*–weak continuous operator, because $U^*(E^*) \subset L_1(\mu \mid \Sigma)$ (identify $U^*(x^*)$ with $d\langle m, x^*\rangle/d\mu \mid \Sigma$).

So, if $A_i \in \Delta$ and $A_i \downarrow \phi$, then $[\chi_{A_i}] \to 0$ weak* in $L_\infty(\mu)$. This implies $\alpha^*([\chi_{A_i}]) \to 0$ weak* in $L_\infty(\mu \mid \Sigma)$, and this implies $U\alpha^*([\chi_{A_i}]) \to 0$ weakly in E.

Thus, $\tilde{m}$ is weakly countably additive and therefore countably additive [1, IV.10.1].

Finally, to see that $\tilde{m}$ is absolutely continuous with respect to μ, it is enough to show that the set of numerical measures $K = \{\langle \tilde{m}, x^*\rangle : x^*$ is in E^* and $\| x^* \| \leq 1\}$ is uniformly absolutely continuous with respect to μ. But, since $\tilde{m}$

is a closed subspace of $bv\kappa a(S, \Sigma_\tau, E)$. Of course, m is absolutely continuous with respect to its variation function.

THEOREM 2.2. *The dual of $bv\kappa a(S, \Sigma_\tau, E)$ is isometrically isomorphic to the substitution space $P_{l_\infty(\mu)}N_\mu{}^*$.*

This follows directly from Theorem 1 and the theory of substitution spaces [8, p. 35].

It should be noted that the decompositions of the type given in Theorems 2.1 and 2.2 were first obtained (to the best of my knowledge) by Artemenko [10]. This decomposition was also obtained by Sreider [11] in a different form and was used by him to study the spectrum of $M(G)$. Also, it follows from Kakutani's M-space theory [12] that $\kappa a^*(S, \Sigma_\tau)$ is isometrically isomorphic to a space $C(K)$, where K is a compact T_2 space. It is possible to construct this space from the decomposition given in Theorem 2.2 as follows: In the case of scalar measures, $N_\mu{}^* = L_\infty(\mu)$. So, let K_α be the Stone space of $L_\infty(\mu)$. Let X be the disjoint union of the spaces K_α. Let $G = \{f \mid f$ is bounded and $f \mid K_\alpha \in C(K_\alpha)\}$. Then $\mathcal{O}$ is a uniformly closed algebra of real-valued functions on X. Let K be the "compactification" of X such that the extension map $f \to \bar{f}$ takes $\mathcal{O}$ onto $C(K)$ [13]. Then, $\kappa a^*(S, \Sigma_\tau) \cong C(K)$.

Let us consider now the dual of spaces of vector-valued measures.

Suppose μ is a probability measure in $\kappa a(S, \Sigma_\tau)$. Let $H(S, \Sigma_\tau, E^*, \mu)$ be the space of all additive functions ν from Σ_τ into E^* for which there is a number α such that $\| \nu(E) \| \leqslant \alpha\mu(E)$, for every $E \in \Sigma$. Also, for each $\nu \in H(S, \Sigma_\tau, E^*, \mu)$, let $\| \nu \| = \sup\{[\| \nu(E)\| / \mu(E)] \mid \mu(E) > 0\}$.

The space $H(S, \Sigma_\tau, E^*, \mu)$ is a Banach space under this norm [3, 9]. Also, if $\nu \in H(S, \Sigma_\tau, E^*, \mu)$, then $\nu \in bv\kappa a(S, \Sigma_\tau, E^*)$.

We will need the following theorem of Uhl [9].

THEOREM 2.3. *Suppose E has the Radon–Nikodym property and μ is a positive measure in $\kappa a(S, \Sigma_\tau)$. Then for each $T \in N_\mu{}^*(S, \Sigma_\tau, E)$ there is only one function ν in $H(S, \Sigma_\tau, E^*, \mu)$ such that*

$$\text{(U)} \quad T(\lambda) = \int_S [(d\lambda)(d\nu)/d\mu]$$

for all λ in $N_\mu(S, \Sigma_\tau, E)$. Moreover, if (U) *holds, then $|T| = \| \nu \|$ and the mapping of $N_\mu{}^*$ into $H(\Sigma, \mu, X)$ defined by* (U) *is onto.*

A proof of this theorem appears in Uhl's paper [9] and in [3, Theorem 5].

The integral appearing in (U) is a Hellinger-type integral. The theory of this integral is developed in [3, 9].

DEFINITION. A function f from Σ_τ into a linear space X is said to be μ-additive, where μ is a nonnegative measure on Σ_τ, provided $f(E_1) + f(E_2) = f(E_1 \cup E_2)$, whenever E_1 and E_2 are disjoint sets in Σ_τ with both $\mu(E_1)$ and $\mu(E_2)$ positive.

DEFINITION. If φ is a selector for $H - \{\mu_0\}$, then $M(S, \Sigma_\tau, \varphi, E^*)$ consists of all functions ψ from Σ_τ into E^* which are bounded and

(1) $\mu_0\psi$ is μ_0-additive on S,

(2) $\mu_\alpha\psi$ is μ_α-additive on $B_\alpha = \varphi(\mu_\alpha)$, for each $\alpha \in \Gamma$,

(3) if $\mu_0(B) = 0$ and there is no $\gamma \in \Gamma$ such that $\mu_\gamma(B) > 0$ and $B \subseteq B_\gamma$, then $\psi(B) = 0$.

The space $M(S, \Sigma_\tau, \varphi, E^*)$ is a Banach space under the uniform norm.

THEOREM 2.4. *Suppose there is a maximal collection of mutually singular positive measure $H = \{\mu_0\} \cup \{\mu_\gamma : \gamma \in \Gamma\}$ in $\kappa a(S, \Sigma_\tau)$ such that there is a selector for $H - \{\mu_0\}$. If E has the Radon–Nikodym property, then for each $T \in bv\kappa a^*(S, \Sigma_\tau, E)$ there is only one function ψ in $M(S, \Sigma_\tau, \varphi, E^*)$ such that*

$$\text{(R)} \quad T(\omega) = \int_S \psi \, d\omega$$

for all ω in $bv\kappa a(S, \Sigma_\tau, E)$. Moreover, if (R) *holds, then $|T| = \|\psi\|$ and the spaces $M(S, \Sigma_\tau, \varphi, E^*)$ and $bv\kappa a^*(S, \Sigma_\tau, E)$ are isometrically isomorphic via the mapping defined by* (R).

This is [3, Theorem 7].

There is a simple argument to show the existence of a selector on certain measure spaces.

THEOREM 2.5. *Suppose H is a maximal collection of mutually singular measures from $\kappa a(S, \Sigma_\kappa)$ and $\mu_0 \in H$. If $|H| \leqslant \kappa$, then there is a selector for $H - \{\mu_0\}$.*

Proof. Let $H - \{\mu_0\}$ be well ordered into an initial type (we start with the ordinal 1):

$$H - \{\mu_0\} = \mu_1, \mu_2, \mu_3, \ldots, \mu_\gamma, \ldots, \gamma < \omega_\tau, \omega_\tau \leqslant \kappa.$$

For each γ and α, $0 \leqslant \gamma < \alpha < \omega_\tau$, let $B_{\gamma\alpha}$ be a set in Σ_κ such that $\mu_\gamma(B_{\gamma\alpha}) = 0$ and $\mu_\alpha(B_{\gamma\alpha}') = 0$. For each α, $1 < \alpha < \omega_\tau$, let $B_\alpha = \bigcap_{\gamma<\alpha} B_{\gamma\alpha}$. Since each proper initial segment of ω_τ has cardinality less than κ we have

$$\mu_\gamma(B_\alpha) = 0 \qquad \text{and} \qquad \mu_\alpha(B_\alpha') = 0$$

if $\gamma < \alpha$. Let φ be defined by $\varphi(\mu_\alpha) = B_\alpha$, $0 < \alpha < \omega_\tau$. It follows that φ is a selector for $H - \{\mu_0\}$.

In particular,

THEOREM 2.6. *The continuum hypothesis implies that there is a selector for* $\omega_1 a(I, \Sigma)$.

3. THE EFFECT OF SOME AXIOMS

In [3], the continuum hypothesis was used to obtain the representation (U) provided $|\omega_1 a(S, \Sigma)| \leqslant c = \omega_1$. In fact, the cardinality of the space of scalar valued measures is the only restriction in case CH is assumed as is shown in [3]. We shall see that a similar representation can be obtained under other assumptions, but we restrict S to be standard in these cases.

THEOREM 3.1. *If Martin's axiom holds, then each* $\mu \in \omega_1 a(I, \Sigma)$ *has a unique extension to a measure in* $ca(I, \Sigma_c)$.

This theorem is due to Martin and Solovay. In [14, p. 168], they show that Lebesgue measure is c-additive and consequently the σ-algebra of Lebesgue measurable sets is a c-algebra. In fact the argument given by them can be used to show that if μ is any regular Borel measure in a separable space then the family of all μ-measurable sets forms a c-algebra. Martin and Solovay note this effect on [14, p. 169]. Thus, U, the family of all universally measurable sets is a c-algebra and each Borel measure has an extension to a c-additive measure on U. Notice that $\Sigma_c \subset U$ and if μ_1 and μ_2 are c-additive on U and agree on Σ, then $\mu_1 = \mu_2$.

The next theorem is an easy corollary of Theorem 3.1 by applying a Borel isomorphism of S onto I and noticing that it defines a Σ_c-isomorphism. The theorems given below way extend to a larger class of spaces, for example, the analytic Hausdorff spaces as defined in [32], but the author has not checked them.

THEOREM 3.2. *Assume Martin's axiom. Let S be a standard space, then each measure* $\mu \in \omega_1 a(S, \Sigma)$ *has a unique extension to a c-additive measure on* Σ_c.

In view of Theorem 3.2 and the theorems of Sections 1 and 2 we have the following theorems.

THEOREM 3.3. *Suppose Martin's axiom holds and S is standard. Then*

(1) *if* $m \in \omega_1 a(S, \Sigma, E)$ *and* $m \leqslant \mu$, $\mu \in \omega_1 a(S, \Sigma)$, *then m has a unique extension* $\tilde{m}$ *in* $ca(S, \Sigma_c, E)$ *such that* $\tilde{m} \ll \tilde{\mu}$, *where* $\tilde{\mu}$ *is the unique extension of* μ *in* $ca(S, \Sigma_c)$.

(2) *if* $m \in bv\omega_1 a(S, \Sigma, E)$, *then* m *has a unique extension to a measure* $\tilde{m} \in bvca(S, \Sigma_c, E)$ *and* $\text{var}(m) = \| m \| = \text{var}(\tilde{m}) = \| \tilde{m} \|$.

THEOREM 3.4. *Assume Martin's axiom, E has the Radon–Nikodym property, and S is standard. Then*

(1) $bv\omega_1 a(S, \Sigma, E)$ *is isometrically isomorphic to* $bvca(S, \Sigma_c, E)$ *via the extension operator,*

(2) $bv\omega_1 a^*(S, \Sigma, E) \cong bvca^*(S, \Sigma_c, E)$,

(3) $bvca^*(S, \Sigma_c, E) \cong M(S, \Sigma_c, \varphi, E^*)$ *via the representation* (U).

Thus, we see that Martin's axiom $+ \neg$ CH gives the same type representation as one obtains using CH, however, in order to obtain an integral representation of $\omega_1 a^*(S, \Sigma, E)$ we must first make the identification indicated in part (1).

In the remainder of this section we assume that c is a real-valued measurable cardinal. We assume that there is a free probability measure μ defined on $P(I)$, the family of all subsets of I which is c-additive. Recall that a measure μ is said to be free if $\mu(\{x\}) = 0$, for every x in X. In the next two theorems we show that every nonatomic Borel measure has the maximum number of extensions to $P(I)$.

THEOREM 3.5. *There are* 2^c *nonatomic (free) measures on* $P(I)$.

Proof. For each $A \subset I$ with $|A| = c$, let φ be a 1–1 map between I and A. Define $\mu_A(E) = \mu(\varphi^{-1}(E \cap A))$. Then μ_A is a c-additive measure on $P(I)$. If $|A_1| = |A_2| = c$ and $|A_1 \cap A_2| < c$, then $\mu_{A_1}(A_2) = 0$ and $\mu_{A_2}(A_2) = 1$. Thus, if A_1 and A_2 are almost disjoint, then $\mu_{A_1} \neq \mu_{A_2}$. Prikry has shown that if $\tau < c$, then $2^\tau \leqslant c$ [15]. Thus, by Tarski's theorem [29], there are 2^c almost disjoint subsets of I. Then theorem follows. ∎

THEOREM 3.6. *Every nonatomic Borel measure on I has at least* 2^c *extensions to c-additive measures on* $P(I)$.

Proof. If $\nu \in \omega_1 a(I, \Sigma)$ and ν is free, then there are Borel sets N and M, $\mu(N) = 0$, $\nu(M) = 0$, and a Borel isomorphism ψ of $I - N$ with $I - M$ such that if B is a Borel subset of $I - M$, then $\nu(B) = \mu(\psi^{-1}(B))$.

Thus ν can be extended to a c-additive measure defined on all subsets of I.

If ν_1 and ν_2 are two nonatomic Borel measures on I, then by using the isomorphism map described above it follows that ν_1 and ν_2 have the same number of extensions to members of $ca(I, P(I))$.

Since there are only c nonatomic Borel measures on I, the theorem follows. ∎

Once again, we find that each Borel measure on I has a unique extension to a c-additive measure on Σ_c. Thus, we have the following theorem.

THEOREM 3.7. *Theorems 3.1, 3.2, 3.3, and 3.4 all hold, if Martin's axiom is replaced by the hypothesis that c is a real-valued measurable cardinal.*

Also, in accordance with the theorems of Section 1, let us mention

THEOREM 3.8. *If m is s-bounded on Σ, the Borel subsets of I, then m has an s-bounded extension to all subsets of I.*

We would like to point out the special role played by the Borel sets by the following example.

THEOREM 3.9. There is a countably generated and separated σ-algebra of subsets of the unit interval, $\mathcal{A}$, and a free countably additive probability measure, μ, defined on $\mathcal{A}$ which cannot be extended to be a countably additive measure defined on all subsets of I. This can be argued as follows. First, if there is no real-valued measurable cardinal $\kappa \leqslant c$, then Lebesgue measure on the Borel sets is such an example. Otherwise, let κ be the least real-valued measurable cardinal $\leqslant c$. Let $X \subset I$ with $|X| < \kappa$ and X has outer Lebesgue measure 1. The existence of such a set was proven by Kunen in his thesis [27]. Let $\mathcal{A} = \{E = (X \cap B) \cup (X' \cap C)$: B, C are Borel subsets of $I\}$. Then $\mathcal{A}$ is a countably generated and separated σ-algebra of subsets of I. Let G be a Borel set containing X and define $\mu(E) = \lambda(G \cap B)$, where $E = (X \cap B) \cup (X' \cap C)$. It follows that μ is a countably additive probability measure which agrees with Lebesgue measure λ on the Borel sets. If μ could be extended to all subsets of I, then μ would be κ-additive and thus $\mu(X) = 0 \neq \mu(X)$.

Example 3.9 is due to Kunen who described it in a letter.

It should also be pointed out that E. R. Fisher has carried out a study in his dissertation [17] of the effects of the continuum being a real-valued measurable cardinal. Some implications of various set-theoretical assumptions may be found in [14, 17, 18, 27].

4. GENERATION SCHEMES

In this section, we consider a generation scheme for the c-algebra generated by a family of subsets of a set X. We give some sufficient conditions under which new sets are continually generated by this scheme. The method of proof involves the construction of universal sets.

Let $\mathscr{F}$ be a family of subsets of a set X. We may generate $\mathcal{A}_c(\mathscr{F})$, the smallest c-algebra containing $\mathscr{F}$ by successively closing $\mathscr{F}$ with respect to the complement operator, C, and the operator M_λ, which maps $(P(X))^\lambda \to P(X)$, by mapping a point of $(P(X))^\lambda$ to the intersection of its coordinates.

We shall follow the classical procedure for generating $\mathcal{A}_{\omega_1}(\mathscr{F})$:

Let $\mathscr{F} = Z_0$ and for each ordinal α, $\alpha < \omega_1$, let $Z_\alpha = M_\omega((\bigcup_{\gamma<\alpha} CZ_\gamma)^\omega)$. Thus, we have the following criss-cross diagram:

$$\begin{array}{ccccccc} \mathscr{F} = Z_0 & & Z_1 & & Z_2 & & Z_\alpha \\ \big\downarrow M_\omega & \nearrow & \big\downarrow M_\omega & \nearrow & \cdots & & \big\downarrow \quad \cdots \\ CZ_0 & & CZ_1 & & CZ_2 & & CZ_\alpha \end{array}$$

Then $\bigcup_{\gamma<\omega_1} Z_\gamma$ is $\mathcal{O}\!\ell_{\omega_1}(\mathscr{F})$, the σ-algebra generated by $\mathscr{F}$. In order to generate $\mathcal{O}\!\ell_c(\mathscr{F})$ we continue as follows. For each $\alpha < c$, $\omega_1 \leqslant \alpha < c$, let

$$Z_\alpha = \bigcup_{\gamma<\alpha} Z_\gamma, \qquad \text{if } \alpha \text{ is a cardinal,}$$

and if $\kappa < \alpha < \kappa^+$, where κ is a cardinal, let

$$Z_\alpha = M_\kappa\left(\left(\bigcup_{\gamma<\alpha} CZ_\gamma\right)^\kappa\right).$$

Thus, in the second case, Z_α is the family of all sets which are the union of κ sets taken from $\bigcup_{\gamma<\alpha} CZ_\gamma$.

In case κ is a cardinal, $\kappa < c$, then $Z_{\kappa^+} = \mathcal{O}\!\ell_{\kappa^+}(\mathscr{F})$. Also, it is clear that $\bigcup_{\alpha<c} Z_\alpha$ is the c-algebra generated by $\mathscr{F}$. The problem discuused here is whether all these iterations are necessary. We give some sufficient conditions in order that all these iterations are necessary in case the family $\mathscr{F}$ is the family of all closed subsets of a standard space. It is of course only necessary to argue this for the unit interval.

THEOREM 4.1. *Suppose that for each cardinal $\lambda < c$, there is an $\mathcal{O}\!\ell_{\lambda^+}(\Sigma)$-measurable map, g_λ, from I onto I^λ. Then, for each α, $\omega_1 \leqslant \alpha < c$ and $\kappa \leqslant \alpha < \kappa^+$, there is a subset G_α of $I \times I$, such that $G_\alpha \in \mathcal{O}\!\ell_{\kappa^+}(\Sigma \times \Sigma)$, and if $F \in Z_\alpha$, then there is some $x \in I$ such that F is the section of G_α over x. $F = \{y: (x, y) \in G_\alpha\}$.*

Proof. The proof proceeds by transfinite induction. Let G_{ω_1} be an analytic subset of the unit square which is universal for all analytic subsets of I [16, p. 253]. G_{ω_1} has the required properties for $\alpha = \omega_1$.

Suppose $\omega_1 \leqslant \alpha < c$ and the sets G_γ have been constructed for each ordinal γ, $\gamma < \alpha$.

Let $H_\gamma = G_\gamma'$.

Case I. $\kappa \leqslant \gamma < \gamma + 1 = \alpha < \kappa^+$.

Let g_κ be given by its coordinate functions $g_\kappa = (g_\kappa^1, g_\kappa^2, \dots, g_\kappa^\sigma, \dots)$ $\sigma < \kappa$. For each $\sigma < \kappa$, define $T_\sigma : I \times I \to I \times I$ by $T_\sigma(z, \eta) = (g_\kappa^\sigma(z), \eta)$. It follows that for each $\sigma < \kappa$, g_κ^σ is an $\mathcal{O}\!\ell_{\kappa^+}(\Sigma)$-measurable function and T_σ is an $\mathcal{O}\!\ell_{\kappa^+}(\Sigma\times\Sigma)$-measurable mapping of the square.

Thus $T_\sigma^{-1}(H_\gamma)$ is an $\mathcal{O}_{\kappa^+}(\Sigma \times \Sigma)$-measurable subset of $I \times I$ and $G_\alpha = \bigcup_{\sigma<\kappa} T_\sigma^{-1}(H_\gamma)$ is an $\mathcal{O}_{\kappa^+}(\Sigma \times \Sigma)$-measurable subset of $I \times I$.

Now, suppose $F \in Z_\alpha = M_\kappa((CZ_\gamma)^\kappa)$.

It should be noted here that $Z_\sigma \subseteq Z_\rho$, if $\omega_1 \leqslant \sigma < \rho$ and that this relation holds actually for all ordinals $\sigma, \rho, 0 \leqslant \sigma < \rho$.

Let $F = \bigcup_{\sigma<\kappa} A_\sigma$ where $A_\sigma \in CZ_\gamma$. For each $\sigma < \kappa$, let x_σ be a point of I such that the section of H_γ over x_σ is A_σ.

Let x be a point of I such that $g_\kappa{}^\sigma(x) = x_\sigma$. It can be checked that the section of G_α over x is F.

Case II. $\kappa \leqslant \alpha < \kappa^+$ and α is a limit ordinal. Let f map κ onto α and let $G_\alpha = \bigcup_{\sigma<\kappa} T_\sigma^{-1}(H_{f(\sigma)})$.

Clearly, G_α is an $\mathcal{O}_{\kappa^+}(\Sigma \times \Sigma)$-measurable subset of $I \times I$.

If $F \in Z_\alpha$, then F can be expressed as $\bigcup_{\sigma<\kappa} A_\sigma$, where $A_\sigma \in CZ_{f(\sigma)}$. Let x be a point of I such that for each $\sigma < \kappa$, the section of $H_{f(\sigma)}$ over $g_\kappa{}^\sigma(x)$ is A_σ.

It follows that G_α is universal.

THEOREM 4.2. *If for each $\alpha < c$, there is a subset G_α of $I \times I$ which is universal for the family Z_α in the sense described in Theorem 4.1, then for each $\alpha < c$, Z_α is a proper subset of $Z_{\alpha+1}$.*

Proof. Suppose $\kappa \leqslant \alpha < \kappa^+$. If $Z_{\alpha+1} = Z_\alpha$, then Z_α would be the κ-algebra generated by Σ.

Let Δ be the main diagonal of $I \times I$ and let $B = G_\alpha \Delta$. Then B is a $\mathcal{O}_{\kappa^+}(\Sigma \times \Sigma)$ set. Let D be the projection of B onto the first I-coordinate. Since the projection map restricted to Δ is a homeomorphism, it follows that D is a $\mathcal{O}_{\kappa^+}(\Sigma)$ set. Let $C = I - D$.

If $z_{\alpha+1} = Z_\alpha$, then $C \in Z_\alpha$. Let x be a point of I such that C is the section of G_α over x.

If $x \in C$, then $(x, x) \in G_\alpha \cap \Delta$ and x would be a point of $D = I - C$.

If $x \notin C$, then $(x, x) \notin G_\alpha \cap \Delta$ and x would be a point of C.

This contradiction proves the theorem.

THEOREM 4.3. *Assume Martin's axiom together with the negation of CH. Then for each cardinal λ, $\lambda < C$, there is an $\mathcal{O}_{\kappa^+}(\Sigma)$-measurable map of I onto I^κ.*

Proof. The proof of this theorem follows from an examination of the argument given by Solovay and Martin in [14] to show that for each $\lambda < c$, $2^\lambda = c$. We consider the space 2^λ under the product topology.

Let $A = \{A_\alpha\}_{\alpha<\lambda}$ be a family of λ almost disjoint infinite subsets of 2^ω. For each $t \in 2^\omega$, let

$$(g(t))(\alpha) = \begin{cases} 0 & \text{if } t \cap A_\alpha \text{ is finite} \\ 1 & \text{if } t \cap A_\alpha \text{ is infinite.} \end{cases}$$

Thus, g maps 2^ω into 2^λ.

Let us consider a typical subbasic set for the topology of 2^λ, $V_{\alpha,0} = \{f \in 2^\lambda \mid f(\alpha) = 0\}$.

Then $g^{-1}(V_{\alpha,0}) = \{t \in 2^\omega \mid t \cap A_\alpha \text{ is finite}\}$.

For each finite subset K of $A\alpha$, let $M_k = \{t \in 2^\omega \mid t \cap A_\alpha = K\}$. Clearly, M_k is closed in 2^ω and $\bigcup M_k = g^{-1}(V_{\alpha,0})$. Thus, $g^{-1}(V_{\alpha,0})$ is an F_σ set and $g^{-1}(V_{\alpha,1})$ is a G_δ set.

Therefore, if U is a basic open set, $U = \bigcap_{i=1}^{n} V_{\alpha_i,\epsilon_i}$, then $g^{-1}(U)$ is an $F_{\sigma\delta}$ set.

Finally, if W is an open subset of 2^λ, then W is the union of no more than λ basic open subsets of 2^λ and $g^{-1}(W)$ is an $\mathcal{O}_{\lambda+}(\Sigma)$ set.

Thus, g is $\mathcal{O}_{\lambda+}(\Sigma)$ measurable.

Theorem 1 of paragraph 3 in [14] states that g is onto.

Let f be a Borel isomorphism of I onto 2^ω. Let h be an $\mathcal{O}_{\lambda+}(\Sigma)$-isomorphism of $2^\lambda = (2^\omega)^\lambda$ onto I^λ.

The map hgf from I onto I^λ has the required properties.

5. Some Applications of Martin's Axiom

In closing, I would like to point out some easy applications of Martin's axiom. The first concerns the notion of a lifting [19]. Let us suppose that (S, Σ, μ) is a measure space and that μ is σ-finite and has no atoms and Σ is separated and countably generated. Let $\mathcal{M}_\mu$ be the σ-algebra of μ-measurable subsets of S. Since $(S, \mathcal{M}_\mu, \mu)$ is complete, there is a lifting of $L^\infty(S, \mathcal{M}_\mu, \mu)$ into the space of all bounded μ-measurable functions [19]. Assuming CH, L^∞ has a lifting into $B(S, \Sigma)$ provided S is Polish and μ is a Radon measure [19, p. 182]. In fact, the following theorem holds:

Theorem 5.1. *Let (S, Σ) be Borel isomorphic to a universally measurable subset of I. Assuming* CH, *there is a lifting of $L^\infty(S, \mathcal{M}, \mu)$ into $B(S, \Sigma)$.*

If we weaken our assumption from CH to Martin's axiom, then we have:

Theorem 5.2. *Let (S, Σ) be Borel isomorphic to a universally measurable subset of I. Assuming Martin's axiom, there is a lifting of $L^\infty(S, \mathcal{M}, \mu)$ into $B(S, \Sigma_c)$.*

The proof of Theorems 5.1 and 5.2 follow von Neumann's original argument for Lebesgue measure [28]. There are only some small details to be accounted for. We can and do assume that μ is a probability measure.

First, let us show the existence of a lower density map of the Boolean algebra $\mathcal{M}_\mu/\mathcal{N}_\mu$, where $\mathcal{N}_\mu$ is the ideal of all sets of μ-measure zero, into the Boolean algebra Σ; in other words, a map $\theta\colon \mathcal{M}/\mathcal{N} \to \Sigma$ such that

(1) $\theta([A]) \,\Delta\, A \in \mathcal{N}$,
(2) $\theta([S]) = S$, $\theta([\phi]) = \phi$, and
(3) $\theta([A] \wedge [B]) = \theta([A]) \cap \theta([B])$.

To this end, let φ be a Borel isomorphism of (S, Σ) with $(T, \mathcal{Q})$ where T is a universally measurable subset of I and $\mathcal{Q}$ is the relative Borel structure on T. Define $m(A) = \mu(\varphi^{-1}(A))$, for each Borel subset of T. It is easy to check that a subset E of T is m-measurable if and only if $\varphi^{-1}(E)$ is μ-measurable and if E is m-measurable, then $m(E) = \mu(\varphi^{-1}(E))$. This implies that if there is a lower density map of $\mathcal{M}_m/\mathcal{N}_m$ into $\mathcal{Q}$, then there is a lower density map of $\mathcal{M}_\mu/\mathcal{N}_\mu$ into Σ.

Let us define $\bar{m}(B) = m(B \cap T)$ for each Borel subset B of I. Then $\bar{m}$ is a free probability measure on the Borel subsets of I. Since T is universally measurable, it follows that if $A \in \mathcal{Q}$, then A is universally measurable and $\bar{m}(A) = m(A)$. Finally, if A is m-measurable, then A is $\bar{m}$-measurable and $m(A) = \bar{m}(A)$.

Since $\bar{m}$ is a free probability measure defined on the Borel subsets of the unit interval, there a Borel isomorphism τ of I onto I such that $\bar{m}(B) = \lambda(\tau(B))$, for each Borel subset B of I. As before τ defines a measure preserving map between the $\bar{m}$-measurable sets and the λ-measurable subsets of I. Here, of course, λ denotes Lebesgue measure.

Let us define $\bar{\theta}$ from $\mathcal{M}_{\bar{m}}/\mathcal{N}_{\bar{m}}$ into B, the Borel subsets of I by $\bar{\theta}([E]) = \tau^{-1}(\{x \mid \tau(E)$ has lower density 1 at $x\})$. Then $\bar{\theta}$ is a lower density of $\mathcal{M}_{\bar{m}}/\mathcal{N}_{\bar{m}}$ into B.

Now let us define the map ψ from $\mathcal{M}_m/\mathcal{N}_m$ into $\mathcal{M}_{\bar{m}}/\mathcal{N}_{\bar{m}}$ by $\psi([A]) = [B \in \mathcal{M}_{\bar{m}} \mid A \triangle B \in \mathcal{N}_{\bar{m}}]$. It follows that ψ is a Boolean homomorphism of $\mathcal{M}_m/\mathcal{N}_m$ onto $\mathcal{M}_{\bar{m}}/\mathcal{N}_{\bar{m}}$.

So, if we let $\xi[A] = \bar{\theta}(\psi[A]) \cap T$, then ξ is a lower density of $\mathcal{M}_m/\mathcal{N}_m$ into $\mathcal{Q}$. By following the appropriate maps one more time, we find that there is a lower density map θ of $\mathcal{M}_\mu/\mathcal{N}_\mu$ into Σ.

Once we have the existence of a lower density θ, then we may proceed (following von Neumann) by first well ordering $\mathcal{M}_\mu/\mathcal{N}_\mu$ into type c:

$$[A]_0, [A]_1, \dots, [A]_\alpha, \dots, \alpha < c.$$

Second, it is shown by transfinite induction that there is a transfinite sequence of type c of sets in Σ_c:

$$B_0, \dots, B_\alpha, \dots, \alpha < c$$

such that for each $\alpha < c$ and for all choices $\alpha_1, \dots, \alpha_p, \gamma_1, \dots, \gamma_q, \alpha_i, \gamma_i \leqslant \alpha$ the following relation holds:

$$\left(\bigcap_{j=1}^{p} B_{\alpha_j}\right) \cap \left(\bigcap_{j=1}^{q} B'_{\gamma_j}\right) \subset \theta'\left[\left(\bigwedge_{j=1}^{p} [A]_{\alpha_j}\right) \wedge \left(\bigwedge_{j=1}^{q} [A]'_{\gamma_j}\right)\right] \qquad (*)$$

where

$$\theta'([A]) = (\theta[A])'.$$

That there is such of choice of B_α's is the heart of von Neumann's argument. Following von Neumann it can be shown that if the B_τ's, $\tau < \alpha$ have been chosen so that (*) holds, then one may set:

$$B_\alpha = \bigcup \left\{ \theta' \left[\left(\bigwedge_{j=1}^{p} [A_{\alpha_j}] \right) \wedge \left(\bigwedge_{j=1}^{q} [A_{\gamma_j}]' \right) \right] \cap \left(\bigcap_{j=1}^{p} B_{\alpha_j} \right) \cap \left(\bigcap_{j=1}^{q} B_{\gamma_j} \right) \right\}$$

where the union is taken over all positive integers p, q, and $\alpha_1, \dots, \alpha_p, \gamma_1, \dots, \gamma_q$ less than α. Of course, by this method of procedure it follows that $B_\alpha \in \Sigma_c$ (and if CH is assumed, $B_\alpha \in \Sigma$).

It follows that the map φ: $\mathscr{M}/\mathscr{N} \to \Sigma_c$ defined by $\varphi([A]_\alpha) = B_\alpha$ is a lifting of $\mathscr{M}/\mathscr{N}$ into Σ_c and therefore there is a lifting of $L^\infty(S, \mathscr{M}, m)$ into $B(S, \Sigma_c)$. One can check [20] or [28] for details.

So, we have the obvious questions:

Question 5.1. Suppose (S, Σ, m) is a σ-finite measure space. Is there a lower density θ of $\mathscr{M}/\mathscr{N}$ into Σ, if Σ is countably generated?

Question 5.2. Assume Martin's axiom together with the negation of the continuum hypothesis. Under this assumption is there a lifting of $L^\infty(S, \mathscr{M}, m)$ into $B(S, \Sigma)$?

We note:

THEOREM 5.3. *If S is standard, then there is no isometric isomorphism of $B(S, \Sigma_c)$ onto $B(S, \Sigma)$.*

Proof. Assume there is an isometric isomorphism of $B(S, \Sigma_c)$ onto $B(S, \Sigma)$. Let $B(S)$ be the "compactifications" of S such that each $f \in B(S, \Sigma)$ has a unique extension to a continuous function t on $B(S)$ [1, p. 274]. Let $B_c(S)$ be the corresponding "compactification" of S for $B(S, \Sigma_c)$. If follows that there is a homeomorphism φ of $B(S)$ onto $B_c(S)$ such that $T(f) = f \circ \varphi$ is an isometric isomorphism of $C(C_c(S))$ with $C(B(S))$ [1, p. 442].

Next note that if $x \in S$, then $\tilde{\chi}_{\{x\}} \in C(B_c(S))$ and thus $\chi_{\{\sigma^{-1}(x)\}} \in C(B(S))$. Therefore, $\varphi^{-1}(x)$ must actually be a point of S. A similar consideration of φ^{-1} shows that φ maps S onto S.

If E is a Borel subset of S, then $\tilde{\chi}_E \circ \varphi = \chi_{\sigma^{-1}(E)} \in C(B(S))$ and therefore the restriction of φ to S is Borel measurable. Thus, φ is a Borel isomorphism of S onto S such that if $f \in B(S, \Sigma_c)$, then $f \circ \varphi$ is Borel measurable.

However, since S is an uncountable standard space, there is an analytic non-Borel subset of S, A. Then $\varphi(A)$ is analytic and $\xi_{(A)} \in B(S, \Sigma_c)$ since every analytic set is the union of $\aleph_1$ Borel sets. Then $\xi_{\sigma(A)} \circ \varphi = \xi_A$ is Borel measurable. This contradiction proves the theorem..

I have been unable to answer the following:

Question 5.3. Is there an isometric isomorphism of $B(S, \Sigma_c)$ into $B(S, \Sigma)$?

Recently, Daschiell has studied this type of question for the classical spaces of Baire functions regarded as Banach spaces [21] and these results have been extended by Jayne in [31].

In attempting to solve this question, the author came upon the following curious set:

THEOREM 5.4. *Martin's axiom plus the negation of CH implies the existence of a subset K of I such that $|K| = c$ and every subset of K of cardinality less than c is a G_δ with respect to K.*

Proof. Let H be the family of all subsets of I of cardinality less than c. Since Martin's axiom implies $2^\lambda \leqslant c$ if $\lambda < c$, $|H| = c$. Martin's axiom also implies there is a countable family $G = \{A_n\}_{n=1}^{\infty}$ of subsets of I such that the family H and every Borel subset of I is in the family $G_{\sigma\delta}$. These results may be found in [22].

Let $\varphi(x) = \sum_{p=1}^{\infty} (2/3^p) \chi_{A_p}(x)$. It follows from the properties of the characteristic function of a sequence of sets [23] that the set $K = \varphi(I)$ has the required properties.

It may be noted that the family G has Borel order ω_1 [22, Theorem 12] and consequently there are Borel sets of arbitrarily high class with respect to K.

In [24] a study is made of various problems in classical descriptive set theory employing the characteristic function of a sequence of sets. It is well known that CH implies the existence of subsets of the interval which have Borel orders 1 and 2 [13, p. 443]. It is apparently unknown whether Martin's axiom implies the existence of such sets.

Finally, we note an application of Martin's axiom to bimeasurable functions. Let us recall that a Borel function f mapping a Borel subset, D_f, of separable complete metric space, M_1, into a separable complete metric space M_2 is called bimeasurable if f maps Borel subsets of D_f onto Borel subsets of M_2.

Purves proved that f is bimeasurable if and only if $U(f)$ is countable, where $U(f)$ is the set of all $y \in M_2$ such that $f^{-1}(y)$ is uncountable [26]. Later assuming CH, Darst showed that f is bimeasurable if and only if f maps universally measurable subsets of D_f onto universally measurable subsets of M_2 [27]. We note:

THEOREM 5.5. *Assume Martin's axiom. Then f is bimeasurable if and only if f maps universally measurable subsets of D_f onto universally measurable subsets of M_2.*

Proof. We shall follow the proof given by Darst and preserve his notation. The argument remains unchanged until the last two paragraphs on [26, p. 570].

The completion of the proof depends on showing the existence of a universal null set N in $K \times C$ such that the projection of N onto K is K. It is at this point that Darst uses the continuum hypothesis.

We note that the existence of such a set follows from Martin's axiom.

First Martin's axiom implies the existence of a subset L of the Cantor set C such that $|L| = c$ and L intersects each of first category subset of C in a set of cardinality less than c. This was shown by Kunen in [27, Theorem 14.5].

Let φ be a 1-1 map of L onto K and let $N = \{(\varphi(x), x) \mid x \in C\}$. Clearly, $N \subset K \times C$ and the projection map takes N onto K.

It remains to show that N is a universal null set.

For each nonatomic probability measure, μ, on the Borel subsets of $K \times C$, there is a first category subset F of C such that $\mu(K \times F) = 1$. Since $|W \cap F| < C$, we have $|N \cap (K \times F)| < c$. Also, Martin's axiom implies that sets of cardinality less than c are universal null sets in any standard space. Thus, N is a universal null set and the theorem follows.

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Some Effects of Set-Theoretical Assumptions in Measure Theory

R. DANIEL MAULDIN

Department of Mathematics, University of Florida, Gainesville, Florida 32611

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