PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 59, Number 2, September 1976

THE NONEXISTENCE OF CERTAIN INVARIANT MEASURES

PAUL ERDÖS AND R. DANIEL MAULDIN

ABSTRACT. It is shown that there does not exist an uncountable group G and a nontrivial, σ-finite, countably additive measure defined on all subsets of G which is left-invariant.

The purpose of this note is to resolve a point left unclear in a recent paper of F. Terpe [1] and its review [2]. In [1], F. Terpe shows that the existence of a certain "maximal" integral is equivalent to the existence of a nontrival countably additive σ-finite measure m_I defined on all subsets of the interval $I = [0, 1)$ and invariant under translation mod 1. In his review [2] of this paper, J. C. Oxtoby points out that the proof given there for the nonexistence of such a measure tacitly presupposes that the σ-field $2^I \times 2^I$ of subsets of $I \times I$ generated by generalized rectangles is invariant under the shear map S, where $S(x, y) = (x + y, y)$ and addition is mod 1, and that by a theorem of Iwanik [3] this instance of Weil's measurability condition is satisfied if and only if all subsets of $I \times I$ belong to $2^I \times 2^I$. Thus, Terpe's reasoning actually established the nonexistence of m_I only under the hypothesis $2^{I \times I} = 2^I \times 2^I$. Finally, Oxtoby points out in his review that $2^{I \times I} = 2^I \times 2^I$ is implied by CH, but that CH makes the group argument unnecessary. Oxtoby ends his review by stating that the situation is unclear without CH.

We give a short argument below to show that no such hypothesis is needed.

THEOREM. *Suppose G is an uncountable group and μ is a σ-finite countably additive left-invariant measure defined on all subsets of G. Then μ is trivial.*

PROOF. Let M be a subgroup of G of cardinality $\aleph_1$. Let R be the family of all right cosets of M and let A be a subset of G which intersects each set in R in exactly one point.

Let $\mathcal{H} = \{mA: m \in M\}$. Then $\mathcal{H}$ is a family of $\aleph_1$ disjoint sets covering G and if H_1 and H_2 belong to $\mathcal{H}$, then H_2 is a left translate of H_1.

Let $\{K_n\}_{n=1}^{\infty}$ be a sequence of sets of finite measure covering G. For each n, the sets of the form $K_n \cap H$, where $H \in \mathcal{H}$ form a decomposition of K_n and therefore there are not uncountably many H's with $\mu(K_n \cap H) > 0$.

Thus, there is a set H_0 in $\mathcal{H}$ with $\mu(K_n \cap H_0) = 0$ for each n. Therefore, $\mu(H) = 0$ for all $H \in \mathcal{H}$. This implies that $\aleph_1$ is a real-valued measurable

Received by the editors September 30, 1975 and, in revised form, January 30, 1976.
AMS (MOS) subject classifications (1970). **Primary 28A25, 28A70; Secondary 04A10.**
Key words and phrases. Left-invariant measure, measurable cardinal.

cardinal. But, assuming the axiom of choice (which we are in this paper), it is known that $\aleph_1$ is not measurable [4].

REFERENCES

1. Frank Terpe, *Zur Existenz maximaler translationsaleicher Integrale*, Theory of Sets and Topology (in honor of Felix Hausdorff, 1868–1942), VEB Deutscher Verlag, Berlin, 1972, pp. 495–502. MR **49** #10861.

2. J. C. Oxtoby, Math Reviews **49** #10861.

3. A. Iwanik, *On infinite complete algebras*, Colloq. Math. **29** (1974), 195–199, 307. MR **49** #179.

4. S. M. Ulam, *Zur Masstheorie in der allgemeinen Mengenlehre*, Fund. Math. **16** (1930), 141–150.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF FLORIDA, GAINESVILLE, FLORIDA 32611