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BIJECTIVE PROOFS OF JENSEN'S AND MOHANTY-HANDA'S IDENTITIES

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This article provides bijective proofs of Jensen's identity and a multivariate generalization called Mohanty-Handa's identity. Our proofs employ suitable combinatorial operations on lattice paths.

1. INTRODUCTION

Jensen's identity [**6**], also called Jensen's convolution, is the formula

$$\sum_{m=0}^{n} \binom{x+mz}{m}\binom{y-mz}{n-m} = \sum_{k=0}^{n} \binom{x+y-k}{n-k} z^k, \tag{1}$$

which holds for all $n \in \mathbb{N}$ and all formal indeterminates x, y, z. This formula has a multivariate generalization called *Mohanty-Handa's identity* [**9**], which is stated in equation (3) below. The main purpose of this paper is to provide *bijective* proofs of these identities based on lattice path models. Fairly simple (but non-bijective) proofs of these formulas have been given in [**4**]. Bijective proofs of some related identities are given by Guo in [**5**]. However, to the knowledge of the authors, no bijective proof of Mohanty-Handa's identity has previously appeared in the literature.

Before continuing, we would like to provide some context indicating how Jensen's identity (and its relatives) appear in certain parts of probability. Specifically, the identity is useful in the study of Dirichlet distributions, which arise in Bayesian statistics and in Pólya urn schemes [**1, 8**]. We recall how these distributions are defined. Let $m \geq 2$ and $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_m)$ be a real vector with

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each $\alpha_i > 0$. A random vector $(X_1, \ldots, X_{m-1})$ has a *Dirichlet distribution with parameter* α provided it has a density function $f\big((x_1, \ldots, x_{m-1})|\alpha\big)$ supported on the simplex $S = \Big\{(x_1, \ldots, x_{m-1}) : x_i \geq 0 \text{ for all } i, \text{ and } \sum_{i=1}^{m-1} x_i \leq 1\Big\}$ given by

$$f\big((x_1, \ldots, x_{m-1})|\alpha\big) = c(\alpha)\left(\prod_{i=1}^{m-1} x_i^{\alpha_i - 1}\right)\left(1 - \sum_{i=1}^{m-1} x_i\right)^{\alpha_m - 1}.$$

The normalizing factor is given by

$$c(\alpha) = \frac{\Gamma\Big(\sum_{i=1}^{m} \alpha_i\Big)}{\prod_{i=1}^{m} \Gamma(\alpha_i)}.$$

In other words,

$$\int_S x_1^{\alpha_1 - 1} \cdots x_{m-1}^{\alpha_{m-1} - 1}\left(1 - \sum_{i=1}^{m-1} x_i\right)^{\alpha_m - 1} \mathrm{d}x_1 \cdots \mathrm{d}x_{m-1} = \frac{\prod_{i=1}^{m} \Gamma(\alpha_i)}{\Gamma\Big(\sum_{i=1}^{m} \alpha_i\Big)}.$$

For a given positive integer n, if we multiply the integrand by 1 expressed as

$$1 = \sum_{n_1 + \cdots + n_m = n} \binom{n}{n_1, \ldots, n_m} x_1^{n_1} \ldots x_{m-1}^{n_{m-1}} \left(1 - \sum_{j=1}^{m-1} x_j\right)^{n_m},$$

integrate term by term, and use the preceding formula, we obtain the identity

$$\sum_{n_1 + \cdots + n_m = n} \binom{n}{n_1, \ldots, n_m} \frac{\prod_{i=1}^{m} \Gamma(\alpha_i + n_i)}{\Gamma\Big(\sum_{i=1}^{m} (\alpha_i + n_i)\Big)} = \frac{\prod_{i=1}^{m} \Gamma(\alpha_i)}{\Gamma\Big(\sum_{i=1}^{m} \alpha_i\Big)}.$$

One can also prove this identity in the case $m = 2$ by an application of Jensen's identity and its relatives. The case $m > 2$ can be established with the multivariate version of Jensen's identity that we study here.

Turning to the proof of Jensen's identity, it will suffice to prove (1) for *positive integers* x, y, z, since both sides of the identity are polynomials in these three variables. In fact, it is enough to prove it for all $x, y, z \in \mathbb{N}^+$ such that $y \geq nz$. To gain intuition, we first prove the special cases of the identity where $z = 0$, $z = 1$, or $z = 2$.

2. PROOF FOR $z = 0$

The $z = 0$ case of Jensen's identity is the Chu-Vandermonde convolution [**2, 10**]:

$$\sum_{m=0}^{n} \binom{x}{m}\binom{y}{n-m} = \binom{x+y}{n}.$$

To introduce our method we prove this using the lattice path model for binomial coefficients, as follows. Given integers $a \le c$ and $b \le d$, a *lattice path* from (a, b) to (c, d) is a succession of unit-length north steps and east steps that start at (a, b) and end at (c, d). By representing each north step as a 1 and each east step as a 0, we can conveniently encode a lattice path as a word of length $(c+d)-(a+b)$ consisting of $c - a$ zeroes and $d - b$ ones. Using this encoding, it is clear that the number of lattice paths from (a, b) to (c, d) is $\binom{(c+d)-(a+b)}{c-a} = \binom{(c+d)-(a+b)}{d-b} = \binom{(c+d)-(a+b)}{c-a,\, d-b}$. It follows that the left side of the identity counts lattice paths that first take x steps from $(0, 0)$ to $(x - m, m)$ (for some m between 0 and n) and then take y steps from $(x-m, m)$ to $(x+y-n, n)$. But these are exactly the lattice paths from $(0, 0)$ to $(x + y - n, n)$. We see that the left side is classifying these lattice paths based on where they intersect the line $X + Y = x$ (see Figure 1).

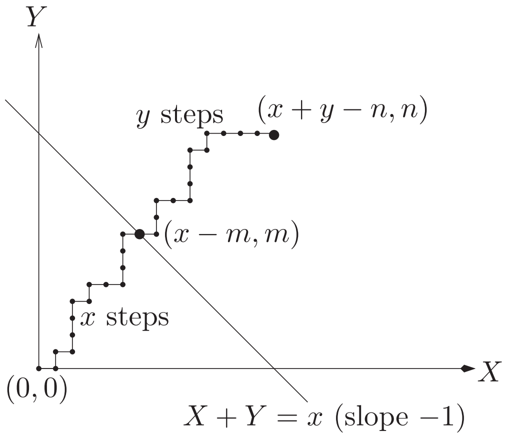

Figure 1. Proof of Jensen's identity when $z = 0$

3. PROOF FOR $z = 1$

The $z = 1$ case of Jensen's identity says

$$\sum_{m=0}^{n} \binom{x+m}{m}\binom{y-m}{n-m} = \sum_{k=0}^{n} \binom{x+y-k}{n-k}. \tag{2}$$

The left side counts all triples (m, P_1, P_2), where $0 \le m \le n$, P_1 is a lattice path from $(0, 0)$ to (x, m), and P_2 is a lattice path from (x, m) to $(x + y - n, n)$. By concatenating the paths P_1 and P_2, we can identify such a triple with a single lattice path P from $(0,0)$ to $(x+y-n,n)$ with one *marked* lattice point on the vertical line $X=x$. The marker indicates where P_1 ends and P_2 begins. Let A denote the set of all such paths.

The right side of (2) counts the set B of all lattice paths Q from $(0,0)$ to $(x+y-n,n-k)$, for some choice of k between 0 and n. To prove (2), it suffices to exhibit a bijection $f: A \to B$. Given a marked path $P \in A$, suppose that P takes k unit-length vertical steps on the line $X=x$ before reaching the marker. Erase these k steps to obtain $Q = f(P) \in B$. The inverse of f acts on any $Q \in B$ as follows. Since n is fixed and known, we recover k by noting the final y-coordinate $n-k$ of the path Q. To obtain $P = f^{-1}(Q) \in A$, follow the path Q until it first reaches the line $X=x$. Splice in k new vertical steps here, and mark the next vertex on the path. See Figure 2.

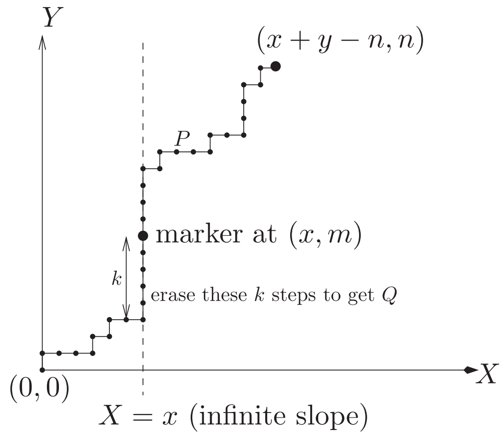

Figure 2. Proof of Jensen's identity when $z=1$

4. COMBINATORIAL INTERPRETATION FOR $z \geq 2$

Now we study the case where $z \geq 2$ is an integer. The first step is to give combinatorial interpretations of both sides of (1). The left side $\sum_{m=0}^{n} \binom{x+mz}{m}\binom{y-mz}{n-m}$ counts triples (m, P_1, P_2), where $0 \leq m \leq n$, P_1 is a lattice path from $(0,0)$ to $(x+mz-m, m)$, and P_2 is a lattice path from $(x+mz-m,m)$ to $(x+y-n,n)$. As in the case $z=1$, we can identify such a triple with a single *marked* lattice path P from $(0,0)$ to $(x+y-n,n)$, where the marked vertex is required to be a lattice point (i.e., a point with integer coordinates) lying on the line ℓ with equation $Y = \frac{1}{z-1}(X-x)$. See Figure 3 for an illustration in the case $z=3$. Let A be the set of all such marked paths. Our previous restriction to parameter values satisfying $y \geq nz$ ensures that *all* lattice paths from $(0,0)$ to $(x+y-n,n)$ must touch ℓ at one or more lattice points.

Now we describe a set of objects counted by the right side $\sum_{k=0}^{n} \binom{x+y-k}{n-k} z^k$ of (1). Let B be the set of triples $(k, Q, a_1 a_2 \cdots a_k)$, where $0 \leq k \leq n$, Q is a lattice path from $(0,0)$ to $(x+y-n, n-k)$, and $0 \leq a_i < z$ for $1 \leq i \leq k$. To prove Jensen's identity, we need to construct a bijection $f: A \to B$. The basic idea for how f works is the following. Given $P \in A$, suppose $P \cap \ell$ contains k lattice points that precede the marked lattice point on P. Then $f(P)$ will be some triple $(k, Q, a_1 \cdots a_k)$ where Q is a lattice path with k fewer vertical steps than

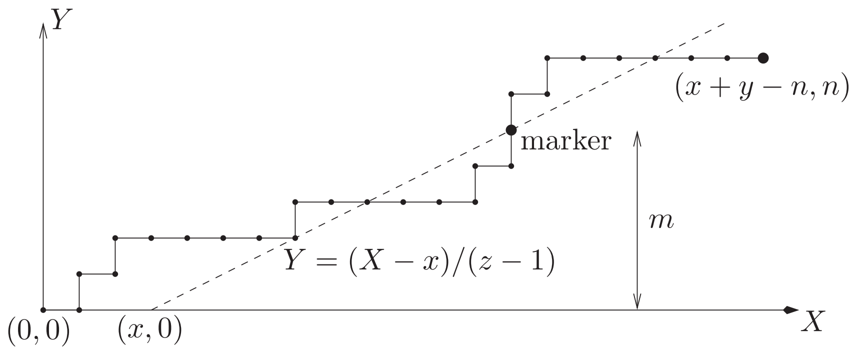

Figure 3. Combinatorial interpretation of the left side of Jensen's identity.

P. The information recorded in the word $a_1 \cdots a_k$ must allow us to reconstruct P from k and Q. Intuitively, we will remove one vertical step from P between each pair of consecutive visits to ℓ before the marker (although other changes will occur as well), and we will also record a value a_i to help us reverse the changes to this portion of P. The precise details of this process are rather intricate, so we begin by considering the somewhat simpler case $z = 2$.

5. PROOF FOR $z = 2$

Keep the notation of the previous section. For any lattice points (a, b) and (c, d), we say that (a, b) lies *weakly below* (c, d), and (c, d) lies *weakly above* (a, b), iff $b \le d$. Given $P \in A$, we describe how to compute $f(P) \in B$ assuming $z = 2$ and $y \ge 2n$. Let the lattice points in $P \cap \ell$ lying weakly below the marked point be $v_0, v_1, \ldots, v_k$, ordered by increasing y-coordinate (note $0 \le k \le n$). Dissect the path P into the concatenation of paths $P_0, P_1, \ldots, P_k, R$, where P_0 goes from $(0, 0)$ to v_0, P_i goes from v_{i-1} to v_i for $1 \le i \le k$, and R goes from the marked point v_k to $(x + y - n, n)$. As in §2, we identify lattice paths with sequences of ones (north steps) and zeroes (east steps). Define modified paths $P'_1, \ldots, P'_k$ as follows. If P_i begins with a 1, erase this 1 to obtain P'_i and set $a_i = 1$. If P_i begins with a 0, obtain P'_i by replacing every step s in P_i by $1-s$ (i.e., interchange the roles of north steps and east steps) and then erasing the initial 1; also set $a_i = 0$. Then define $f(P) = (k, Q, a_1 a_2 \cdots a_k)$ where Q is the concatenation of $P_0, P'_1, P'_2, \ldots, P'_k, R$.

For example, suppose $x = 2$, $y = 20$, $z = 2$, $n = 9$, and P is the path shown in Figure 4. Then $k = 4$, $P_0 = 00$, $P_1 = 110100$, $P_2 = 01$, $P_3 = 0011$, $P_4 = 10$, and $R = 110000$. We calculate $P'_1 = 10100$, $P'_2 = 0$, $P'_3 = 100$, $P'_4 = 0$, and hence

$$f(P) = (4, 00\,10100\,0\,100\,0\,110000, 1001).$$

In the example, note that for $0 \le i \le k$, the prefix $P_0 P'_1 \cdots P'_i$ of Q ends at the first lattice point on Q that touches the line $Y = X - x - i$. It is routine to check using the definition of f that this property holds in general. This provides the key to defining a function $g : B \to A$ that will be the two-sided inverse of f. Fix $(k, Q, a_1 \cdots a_k) \in B$. Factor Q into the concatenation of paths $Q_0, Q_1, \cdots, Q_k, R$, where Q_0 starts at $(0,0)$, and each Q_i (for $0 \le i \le k$) ends at the first lattice point on Q and on the line $Y = X - x - i$. Our assumptions on x, y, z, n (namely, $z = 2$, $x, y, n \in \mathbb{N}^+$, and $y \ge 2n$) guarantee that such lattice points do exist. For $1 \le i \le k$, let $Q_i' = a_i Q_i$ if $a_i = 1$. If $a_i = 0$, form Q_i' by interchanging all 0's and 1's in Q_i and then adding a new zero to the beginning. Set

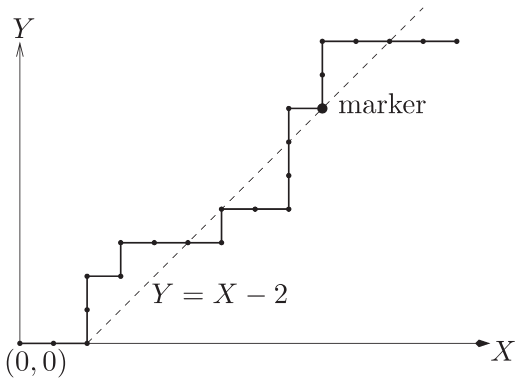

Figure 4. A marked path in A when $z = 2$.

$$g(k, Q, a_1 \cdots a_k) = Q_0 Q_1' Q_2' \cdots Q_k'^{\,*} R,$$

where the asterisk indicates the position of the marker. It is not difficult to verify that $f \circ g = \mathrm{id}_B$ and $g \circ f = \mathrm{id}_A$, so that f and g are both bijections.

As an example of the map g, take $x = 2$, $y = 20$, $z = 2$, and $n = 9$ as before. Suppose $k = 4$, $Q = 100011000011000000$, and $a_1 \cdots a_4 = 0101$. We calculate $Q_0 = 1000$, $Q_1 = 11000$, $Q_2 = 0$, $Q_3 = 11000$, $Q_4 = 0$, $R = 00$, and hence

$$g(k, Q, a_1 a_2 a_3 a_4) = 1000\,000111\,10\,000111\,10^*\,00.$$

6. BIJECTIVE PROOF FOR $z \ge 2$

In the definition of the map f for $z = 2$, the passage from P_i to P_i' when P_i starts with an east step amounts to *reflecting* the path P_i through the line ℓ and then deleting the first step. When we try to generalize the map to $z > 2$, reflection no longer has the correct effect on the path dimensions. However, *rotational* symmetry is still present, and this will lead to a map with the required properties. A similar idea was used in [**3**] and [**7**], which discuss a generalization of "André's reflection principle."

We are now ready to define the map $f : A \to B$ for $z \ge 2$. (When $z = 2$, this will be a different map from the one discussed in the previous section.) As before, given $P \in A$, let $v_0, v_1, \ldots, v_k$ be the lattice points in $P \cap \ell$ lying weakly below the marked point, ordered by increasing y-coordinate. We write P as the concatenation of paths $P_0, P_1, \ldots, P_k, R$, where P_0 goes from $(0,0)$ to v_0, P_i goes from v_{i-1} to v_i for $1 \le i \le k$, and R goes from the marked point v_k to $(x + y - n, n)$.

Define the *level* of any lattice point (a, b) to be $\mathrm{lv}(a, b) = (z-1)b - a + x$. Note that points on ℓ have level zero; taking one north step increases the level by $z-1$; and

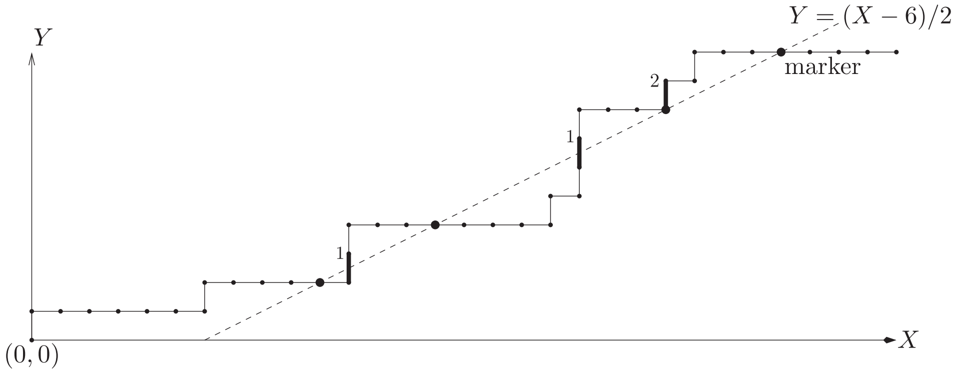

Figure 5. A marked path in A when $z = 3$.

taking one east step decreases the level by 1. For $1 \le i \le k$, we transform P_i into a new path P_i' as follows. Look for a lattice point $(a, b) \neq v_i$ on P_i such that $\mathrm{lv}(a, b)$ is ≤ 0 and the level of the next point on P_i is some value $j \ge 0$. Such a lattice point must exist, since P_i is a lattice path of positive length that begins and ends at level zero. Since levels decrease by at most one at every step, the lattice point (a, b) is unique (as v_i is the only point following (a, b) on P_i that can have a non-positive label). Note that the next point after (a, b) must be $(a, b+1)$ in this situation. To obtain P_i' from P_i, take the reversal of the steps leading from v_{i-1} to (a, b), followed by the steps leading from $(a, b+1)$ to v_i. Set $a_i = j$ (noting $0 \le j < z$), and let Q be the concatenation of $P_0, P_1', \ldots, P_k', R$. Finally, let $f(P) = (k, Q, a_1 \cdots a_k)$.

For example, suppose $x = 6$, $y = 34$, $n = 10$, $z = 3$, and the path P is

100000010000011000000010111000101000*0000,

as shown in Figure 5. Here $k = 3$, $P_0 = 100000010000$, $P_1 = 011000$, $P_2 = 000010111000$, $P_3 = 101000$, and $R = 0000$. Between every two consecutive visits to ℓ before the marker, we look for the first north step that goes from a point weakly below ℓ to a point weakly above ℓ. (A point (a, b) is *weakly below* a line $y = cx + d$ iff $b \le ca + d$; similarly, (a, b) is *weakly above* this line iff $b \ge ca + d$.) See the thick north steps in the figure, where the label of the point at the end of each such step is also shown. We compute $P_1' = 01000$, $a_1 = 1$, $P_2' = 10100001000$, $a_2 = 1$, $P_3' = 01000$, $a_3 = 2$, and finally

$$f(P) = (3, P_0P_1'P_2'P_3'R, 112).$$

Before describing the inverse map $g : B \to A$, let us discuss how we may recover P_i from P_i' and a_i. Recall that P_i goes from level zero through strictly negative levels to the point (a, b) at level $a_i - (z-1) \in \{0, -1, \ldots, -(z-1)\}$, then takes a "critical north step" to level a_i, and then returns to level zero through strictly positive levels. Suppose momentarily that P_i' (as defined above) starts at level zero. Write $u = |a_i - (z-1)|$ for convenience. Let the levels visited by P_i before the critical north step be $l_0 = 0, l_1, l_2, \ldots, l_s = -u$. One readily verifies that the levels first visited by P_i' are $-u-l_s = 0, -u-l_{s-1}, \dots, -u-l_1, -u-l_0 = -u$. In particular, we can recover the location of the deleted north step by finding the first point visited by P_i' with label $-u$. In $Q = f(P)$, P_1' does start at level zero. But because a north step was deleted in the passage from P_1 to P_1', P_2' starts at level $-(z-1)$. Similarly, P_i' starts at level $-(i-1)(z-1)$ in Q. So the prescription for finding the location of the deleted north step in P_i' must be modified accordingly.

Here is an algorithm for computing $P = g(k, Q, a_1 \cdots a_k)$. Follow the path Q from $(0,0)$ until it first touches ℓ at a lattice point v_0. If $k = 0$, mark this vertex and continue along Q to complete the path P. If $k > 0$, scan ahead from v_0 to the first lattice point at level $a_1 - (z-1)$; reverse the part of Q from v_0 to this point; splice in a new north step; and continue following Q until the next visit to a lattice point v_1 on ℓ. (Note that v_1 does not lie on ℓ until after the new north step is inserted.) Continue scanning in this way until $a_1, \dots, a_k$ have all been used; then mark the current point on ℓ and copy the rest of Q to complete the path P. Using the comments in the previous paragraph, it is not difficult to verify that g reverses the action of f. So f and g are both bijections, completing the proof of Jensen's identity.

For example, take $(x, y, z, n) = (6, 34, 3, 10)$, $k = 3$,

$$Q = 1000000100000010001010000010000010000000,$$

and $a_1 a_2 a_3 = 201$. We find that $g(k, Q, 201)$ is

$$P = 1000000100000\underline{1}01000000010000101\underline{1}0\underline{1}1000^*0000,$$

where the new north steps have been underlined.

7. PROOF OF MOHANTY-HANDA'S IDENTITY

This section generalizes the preceding proof of Jensen's identity to a bijective proof of a multivariable generalization of this identity due to Mohanty and Handa [**9**]. To state the identity, we need some notation. Fix $d \in \mathbb{N}^+$ and $\mathbf{n} = (n_1, \dots, n_d) \in \mathbb{N}^d$. Let $x, y, z_1, \dots, z_d$ be formal indeterminates, and let $\mathbf{z} = (z_1, \dots, z_d)$. Boldface summation variables $\mathbf{m}$ and $\mathbf{k}$ will range over values in $\mathbb{N}^d$. For $\mathbf{k} = (k_1, \dots, k_d)$, write $|\mathbf{k}| = k_1 + \cdots + k_d$. For any formal polynomial u, we use the notation

$$\binom{u}{\mathbf{m}} = \frac{u(u-1)(u-2)\cdots(u-|\mathbf{m}|+1)}{m_1! m_2! \cdots m_d!},$$

which reduces to a multinomial coefficient when u is a sufficiently large positive integer. Write $\mathbf{z}^{\mathbf{k}} = z_1^{k_1} \cdots z_d^{k_d}$. A summation symbol $\sum_{\mathbf{m}=\mathbf{0}}^{\mathbf{n}}$ means we should sum over all $\mathbf{m} \in \mathbb{N}^d$ with $0 \le m_i \le n_i$ for $1 \le i \le d$. Finally, we use the usual notation for vector addition, vector subtraction, and the dot product of two vectors. The Mohanty-Handa identity states that for all $x, y, d, \mathbf{z}, \mathbf{n}$ as above,

$$\sum_{\mathbf{m}=\mathbf{0}}^{\mathbf{n}} \binom{x + \mathbf{m} \bullet \mathbf{z}}{\mathbf{m}} \binom{y - \mathbf{m} \bullet \mathbf{z}}{\mathbf{n} - \mathbf{m}} = \sum_{\mathbf{k}=\mathbf{0}}^{\mathbf{n}} \binom{x + y - |\mathbf{k}|}{\mathbf{n} - \mathbf{k}} \binom{|\mathbf{k}|}{\mathbf{k}} \mathbf{z}^{\mathbf{k}}. \tag{3}$$

This reduces to Jensen's identity (1) when $d = 1$. It suffices to prove the result assuming x, y, and every z_i is a positive integer; we can even assume that all $z_i \geq 2$ and that y is "large."

Our proof follows the same steps used to prove (1). The left side of (3) counts a certain set of marked lattice paths in $(d+1)$-dimensional Euclidean space, as follows. Let the coordinates for this space be denoted by $X_0, X_1, \ldots, X_d$ (in the $d = 1$ case, we took $X_0 = X$ and $X_1 = Y$). A lattice path from $(0, 0, \ldots, 0)$ to $(x + y - |\mathbf{n}|, n_1, n_2, \ldots, n_d)$ can be encoded by a word w consisting of $x + y - |\mathbf{n}|$ zeroes, n_1 ones, etc., where the letter i encodes a unit-length step parallel to the X_i-axis. For a fixed $\mathbf{m}$, the multinomial coefficient $\binom{x+\mathbf{m}\bullet\mathbf{z}}{\mathbf{m}}$ counts paths from the origin to $(x_0, m_1, m_2, \ldots, m_d)$, where x_0 must satisfy $x_0 + m_1 + \cdots + m_d = x + \mathbf{m}\bullet\mathbf{z}$; i.e., $x_0 - x = m_1(z_1 - 1) + \cdots + m_d(z_d - 1)$. Then $\binom{y-\mathbf{m}\bullet\mathbf{z}}{\mathbf{n}-\mathbf{m}}$ counts paths from this point to $(x + y - |\mathbf{n}|, n_1, \ldots, n_d)$. Let H denote the hyperplane in $\mathbb{R}^{d+1}$ with equation

$$X_0 - x = X_1(z_1 - 1) + X_2(z_2 - 1) + \cdots + X_d(z_d - 1).$$

The preceding remarks show that the left side of (3) counts lattice paths P in $\mathbb{R}^{d+1}$ from the origin to $(x+y-|\mathbf{n}|, n_1, \ldots, n_d)$ that touch H at a lattice point and have one lattice point in $H \cap P$ marked. Let A be the set of such marked paths. We assume that y is so large that *every* path P ending at the indicated point must touch the hyperplane H.

The right side of (3) counts objects of the form $(\mathbf{k}, Q, w_0, w_1, \ldots, w_d)$, where: $\mathbf{k} \in \mathbb{N}^d$ and $0 \leq k_i \leq n_i$ for $1 \leq i \leq d$; Q is a lattice path from $\mathbf{0}$ to $(x+y-|\mathbf{n}|, n_1 - k_1, \ldots, n_d - k_d)$; w_0 is a word consisting of k_1 ones, k_2 twos, $\ldots$, k_d d's; and for $1 \leq i \leq d$, w_i is a word of length k_i using letters in $\{0, 1, \ldots, z_i - 1\}$. Let B be the set of such objects; our task is to define mutually inverse functions $f : A \to B$ and $g : B \to A$.

To define f, fix a marked path $P \in A$. Let $v_0, v_1, \ldots, v_k$ be the lattice points in $P \cap H$ up to and including the marked point, indexed in the order in which they are encountered along P. Dissect P into subpaths $P_0, P_1, \ldots, P_k, R$, where P_0 goes from the origin to v_0, P_i goes from v_{i-1} to v_i for $1 \leq i \leq k$, and R goes from the marked point v_k to $(x+y-|\mathbf{n}|, n_1, \ldots, n_d)$. We describe how to compute $f(P) = (\mathbf{k}, Q, w_0, w_1, \ldots, w_d)$. Initialize $\mathbf{k}$ to $\mathbf{0}$ and all words w_j to be the empty word. The path Q will be the concatenation of certain paths $P_0, P'_1, \ldots, P'_k, R$. Define the *level* of $(x_0, x_1, \ldots, x_d)$ to be $(z_1 - 1)x_1 + \cdots + (z_d - 1)x_d - x_0 + x$. Note that lattice points on H have level zero; taking a step in the positive X_i-direction (for $i > 0$) increases the level by $z_i - 1$; and taking a step in the positive X_0-direction decreases the level by 1. Consider a subpath P_j, which is a positive-length lattice path that starts and ends at level 0 with no other visits to level 0. As in the $d = 1$ case, we see that there exists a unique step s in P_j that goes from some level ≤ 0 to some level ≥ 0. We view the step s as an integer in $\{1, 2, \ldots, d\}$ as explained earlier. Process P_i as follows: increment k_s by 1; form P'_i by taking the part of P_i preceding s in reverse, then taking the part of P_i following s in the forward direction; append the letter s to the word w_0; and append the level of the vertex at the end of step s (which lies in $\{0, 1, \ldots, z_s - 1\}$) to the word w_s. It is routine to check that this algorithm will produce an object $f(P) \in B$.

Intuitively, the word w_0 remembers which type of step was deleted between each pair of visits to H, and the words $w_1, \ldots, w_d$ enable us to recover P_j from P'_j as in the $d = 1$ case. Here is a formal description of the inverse map $g : B \to A$. Start with $(\mathbf{k}, Q, w_0, w_1, \ldots, w_d) \in B$, and build the path P as follows. Follow Q until it hits H. Repeatedly do the following until $\mathbf{k}$ becomes zero and all letters in the words w_j are consumed. Let $v \in H$ be the current position on Q. Let s be the next unused letter in w_0, and let ℓ be the next unused letter in w_s. Decrement k_s by 1. Scan ahead from v to the first lattice point at level $\ell - (z_s - 1)$; reverse the part of Q from v to this point; splice in a new step in the X_s-direction; and continue following the newly modified path Q until hitting the next lattice point on H. When $\mathbf{k}$ reaches zero and all words have been used up, mark the current vertex and copy the rest of Q to complete the path P. One may check that this algorithm produces a marked path in A, and that $f \circ g = \mathrm{id}_B$, $g \circ f = \mathrm{id}_A$. This completes the proof.

Pseudocode implementing the algorithms defining f and g appears in the appendix following Section 8.

8. PROOF FOR NEGATIVE z

This section provides another bijective proof of Jensen's identity

$$\sum_{m=0}^{n} \binom{x + mz}{m} \binom{y - mz}{n - m} = \sum_{k=0}^{n} \binom{x + y - k}{n - k} z^k$$

in which we verify the identity for all *negative* integers z. It suffices to prove the result under the additional assumptions that x, y, n are positive integers satisfying $x \geq n|z|$. This second proof is in some ways simpler than the bijection given earlier, but we must now use *signed* objects.

Let ℓ be the line $X - x = Y(z - 1)$ with slope $1/(z - 1)$. As before, the left side of the identity counts the set A of all lattice paths from $(0, 0)$ to $(x + y - n, n)$ that have one lattice point on the line ℓ marked. Since the slope of the line is now negative, every lattice path from $(0, 0)$ to $(x + y - n, n)$ must intersect ℓ in exactly one point (this uses the assumption $x \geq n|z|$). However, this intersection point may not be a *lattice* point with integer coordinates. See Figure 6 for examples in the case $z = -2$. Since the intersection point is unique, we can forget the marker and say that A is the set of all lattice paths from $(0, 0)$ to $(x + y - n, n)$ that intersect the line ℓ at a lattice point.

The right side of Jensen's identity now counts signed objects $(k, P, a_1 \cdots a_k)$ where $0 \leq k \leq n$, P is a lattice path from $(0, 0)$ to $(x + y - n, n - k)$, each $a_i \in \{1, 2, \ldots, |z|\}$, and the sign of the indicated object is $(-1)^k$. Let B be the set of such signed objects. To complete the proof, we will define a sign-reversing involution $I : B \to B$ whose fixed points are in bijective correspondence with the paths in A.

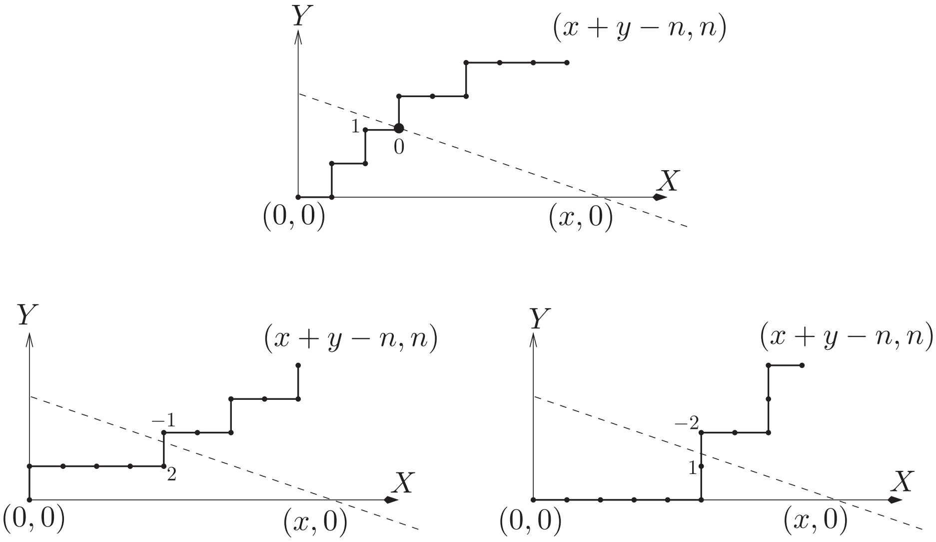

Figure 6. Sample paths for $z = -2$ (only the top path is in A).

As before, we need to label the lattice points on the lattice paths under consideration. We define the labeling so that lattice points on ℓ have label zero; taking one unit-length east step decreases the label by 1; and taking one unit-length north step causes the label to change by $z - 1 \leq -2$. It follows that $(0, 0)$ has label $x \geq n|z|$, and labels strictly decrease as we traverse the path starting at $(0, 0)$. Since paths ending at $(x + y - n, n)$ take $x + y$ total steps, we see that each such path does cross the line ℓ.

First we describe the fixed points of the map I. By definition, these are objects $(k, P, a_1 \cdots a_k)$ in B such that $k = 0$ and P has a vertex labeled zero. Here $a_1 \cdots a_k$ is the empty word, and the vertex labeled zero is a lattice point that lies on both P and ℓ. It follows that these fixed points can be identified with the lattice paths in A.

Now consider a general object $u = (k, P, a_1 \cdots a_k) \in B$. The path P either does or does not have a vertex with label $a_1 + \cdots + a_k$.

Case 1: The label $a_1 + \cdots + a_k$ *does not* appear in P. Note that $0 \leq a_1 + \cdots + a_k \leq k|z| \leq n|z|$, the path starts at a point labeled $n|z|$, and the path ends at a non-positive label. Keeping in mind the definition of the labeling, we see that the only way case 1 occurs is if P takes a north step that "jumps over" the label $a_1 + \cdots + a_k$ (cf. Figure 6). More precisely, there must exist a unique $a_{k+1} \in \{1, 2, \ldots, |z|\}$ such that the label $a_1 + \cdots + a_k + a_{k+1}$ *does* appear in P and is followed immediately by a north step that causes the level to drop by $|z - 1|$. Let Q be obtained from P by deleting this north step. Define $I(u) = (k + 1, Q, a_1 \cdots a_{k+1})$. One checks that $I(u) \in B$ and has the opposite sign as u. In particular, if $k = n$, then Case 1 cannot occur since P cannot have any north steps.

Case 2: The label $a_1 + \cdots + a_k$ *does* appear in P and $k > 0$. Create a path Q by splicing a north step into P just after this label, and define $I(u) =$

$(k-1, Q, a_1 \cdots a_{k-1})$. Since the new north step ends at label $a_1+\cdots+a_k-|z-1| < a_1+\cdots+a_{k-1}$, we see that the label $a_1+\cdots+a_{k-1}$ does *not* appear in Q. It is now routine to verify that $I(u) \in B$ in this case, and that $I(I(u)) = u$ in both cases. Note that if $k = 0$ and 0 does appear in P, case 2 does not apply because u is a fixed point of I.

The involution I pairs off all negative objects in B with some of the positive objects, leaving a set of fixed points in bijection with A. Thus, the second proof of Jensen's identity is complete.

APPENDIX. PSEUDOCODE FOR THE MAPS f AND g.

The following pseudocode (which uses the syntax of the C programming language) implements the bijections $f : A \to B$ and $g : B \to A$ described in §7.

```
#define MAXd 10
#define MAX 100
int d,x,y,z[MAX],n[MAX]; // global; code assumes z[0]=0 and z[i]>1 for i>0.

typedef struct{ int P[MAX],marker; } Aobj; /* object in the set A */
 /* Variable "marker" tells us how many steps on P precede the marker. */

typedef struct{ int k[MAXd],Q[MAX],w[MAXd][MAX]; } Bobj; /* object in B */

Bobj f(Aobj in)
{
  int Qlen,wlen[MAXd],lv[MAX],i,j,r,s;
  Bobj out;

  for (i=1; i<=d; i++) out.k[i]=wlen[i]=0;
  Qlen=wlen[0]=0;
  lv[0]=x; for (i=1; i<=x+y; i++) lv[i]=lv[i-1]+z[in.P[i]]-1;
   /* lv[j] is the level of the lattice point on P after j steps of P. */
  i=0;
  while (lv[i]!=0)
  { i++; Qlen++; out.Q[Qlen]=in.P[i]; } // Copy P0 to Q.
  while (i<in.marker) // lv[i]==0 as each loop iteration starts.
  { j=i; while (lv[j+1]<0) j++;
     // next critical step is step j+1 of P.
    s=in.P[j+1]; out.k[s]++; wlen[0]++; out.w[0][wlen[0]]=s;
    wlen[s]++; out.w[s][wlen[s]]=lv[j+1];
    for (r=j; r>i; r--) { Qlen++; out.Q[Qlen]=in.P[r]; } // reversed part
    i=j+1; // skip critical step
    while (lv[i]!=0)
    { i++; Qlen++; out.Q[Qlen]=in.P[i]; } // copy rest of P_i to Q.
  }
  while (i<x+y) // we've hit marker, so copy R to Q.
  { i++; Qlen++; out.Q[Qlen]=in.P[i]; }
  return out;
```

```
}

Aobj g(Bobj in)
{
  int Qlen,wlen[MAXd],wpos[MAXd],lv[MAX],i,j,r,s,ell,ksum;
  Aobj out;

  for (i=0; i<=d; i++) wpos[i]=0; // current positions in w0,w1,...,wd
  ksum=0; for (i=1; i<=d; i++) ksum+=in.k[i];
  lv[0]=x; for (i=1; i<=x+y-ksum; i++) lv[i]=lv[i-1]+z[in.Q[i]]-1;
    /* compute initial levels of lattice points on Q. */
  i=0;
  while (lv[i]!=0) /* copy Q to P until first hitting H */
  { i++; out.P[i]=in.Q[i]; }
  while (ksum>0)
  { wpos[0]++; s=in.w[0][wpos[0]]; /* s is next unused letter in w0 */
    wpos[s]++; ell=in.w[s][wpos[s]]; /* ell is next unused letter in ws */
    ksum--;
    j=i; while (lv[j]!=(ell-(z[s]-1))) j++;
    for (r=j; r>i; r--) { out.P[i+1+j-r]=in.Q[r]; } // reversal
    for (r=x+y-ksum; r>j; r--)
    { in.Q[r]=in.Q[r-1]; lv[r]=lv[r-1]+z[s]-1; }
    in.Q[j+1]=out.P[j+1]=s;  // splice in new step to P and to Q
    i=j+1;
    while (lv[i]!=0) /* find next visit to H */
    { i++; out.P[i]=in.Q[i]; }
  }
  out.marker=i; // mark current visit to H
  while (i<x+y) // copy rest of Q to P to finish
  { i++; out.P[i]=in.Q[i]; }
  return out;
}
```

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