NON-ISOMORPHIC PROJECTIVE SETS

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It is well known that two Borel subsets of the unit interval are Borel isomorphic, if, and only if, they have the same cardinality. The problem of the existence of analytic, non-Borelian subsets of the unit interval, which are not Borel isomorphic, has not been resolved within ZFC. With the additional assumption of the existence of an uncountable coanalytic set which does not contain a perfect set, it has been shown that there are at least three Borel isomorphism classes of analytic non-Borelian sets [4, 5].

In this paper, the isomorphism classes of projective sets are considered under both Borel isomorphisms and under projective isomorphisms.

Two projective subsets, G and H, of I the unit interval, are said to be Borel isomorphic if there is a bijection, f, of G onto H such that when U is open with respect to H, then $f^{-1}(U)$ is Borelian with respect to G, and when V is open with respect to G, then $f(V)$ is Borelian with respect to H. Thus, G and H are Borel isomorphic, if, and only if, there is a generalised homeomorphism of G onto H [1, p. 374]. The sets G and H are said to be projectively isomorphic if there is a positive integer n and a bijection ϕ of G onto H such that, when U is open with respect to H, then $\phi^{-1}(U)$ is a projective set of class $\sum_n^1$, and when V is open with respect to H, then $\phi(V)$ is a projective set of class $\sum_n^1$.

It can be seen that two projective sets G and H are projectively isomorphic, if, and only if, there is a bijection ϕ of G onto H whose graph is a projective subset of the unit square. This same condition does not hold for Borel isomorphisms; *i.e.*, it may be possible to have a bijection of G onto H whose graph is Borelian in $G \times H$, but ϕ is not a Borel isomorphism of G onto H.

In [3], Sierpiński examined a problem of Kuratowski which may be phrased as follows:

If A and B are projective subsets of the unit interval, each of cardinality $2^{\aleph_0}$, then are A and B projectively isomorphic?

In [3], Sierpiński shows that, if both A and B contain perfect sets, then the answer is yes. Thus, if A and B are uncountable analytic sets, then A and B are projectively isomorphic. However, in [2], Kuratowski showed that the existence of a projective well-ordering of the unit interval into type ω_1 implies the existence of uncountable projective sets which do not contain any perfect set. The existence of such a well-ordering follows from $V = L$ [6] and from the existence of a measurable cardinal [7]. Under this assumption, it is shown in Theorem 1 of this paper, that the answer to Kuratowski's question is yes. Of course, it is enough to show that, if E is an uncountable projective set, then E is projectively isomorphic to the unit interval. This is because the relation of projective isomorphism is an equivalence relation. In fact, if f is a projective isomorphism of A onto B and g a projective isomorphism of B onto C, then $g \circ f$ is a bijection of A onto C and the graph, Γ, of $g \circ f$ is a projective set, since

$$\Gamma = \prod_{13}(H),$$

where
$$H = \{(x, y, z) : y = f(x) \wedge z = g(y)\}.$$

Finally, in Theorem 2 of this paper it is shown that there are $2^{\aleph_0}$ projective subsets of I, no two of which are Borel isomorphic.

Notation and Definitions. Throughout this paper, it is assumed that there has been given a well-ordering of the unit interval, I, into type ω_1 such that the set $W = \{(x, y) : x \prec y\}$ is a projective subset of $I \times I$.

If $x \in I$, $A(x) = \{t : t \prec x\}$.

If $X \subseteq I$, then $p(X)$ denotes the first element of X.

The main diagonal of the product space X^α will always be denoted by D.

For each permutation π of the first n positive integers, M_π denotes the corresponding induced map of X^n onto X^n.

In general, sequences will be denoted by lower case Greek letters and the terms of the sequence by corresponding lower case Latin letters.

THEOREM 1. *If E is a projective subset of I and $|E| = c$, then there is a projective isomorphism of I onto E. Thus, two projective sets are projectively isomorphic, if, and only if, they have the same cardinality.*

Proof. Let H be the subset of $I^\omega \times I^\omega$ consisting of all pairs of sequences (α, β) such that for every pair of positive integers i and j, $a_i \prec a_j$, if, and only if, $b_i \prec b_j$.

For each pair of positive integers (i, j), let
$$K_{ij} = \{(\alpha, \beta) : a_i \prec a_j \text{ and } b_i \succcurlyeq b_j\},$$
and
$$J_{ij} = \{(\alpha, \beta) : a_i \succcurlyeq a_j \text{ and } b_i \prec b_j\}.$$

We have
$$H = (I^\omega \times I^\omega) - \bigcup(K_{ij} \cup J_{ij}).$$

Since
$$K_{12} = [(W \times I^\omega) \times I^\omega] \cap [I^\omega \times \{(I^2 - W) \times I^\omega\}],$$
it follows that K_{12} is a projective subset of $I^\omega \times I^\omega$. Similarly, it follows that all the sets K_{ij} and J_{ij} are projective sets of bounded class and therefore H is a projective subset of $I^\omega \times I^\omega$.

Let C be the subset of $I \times I^\omega \times E^\omega \times E$ to which (x, σ, τ, e) belongs, if, and only if, (1) σ is an enumeration of $A(x)$, (2) τ is an enumeration of $B(e) = A(e) \cap E$. (Here an enumeration is not necessarily a one-to-one listing.)

Let ϕ be the projection of $C \cap (I \times H \times I)$ onto its first and last coordinates. Thus, $\phi \subseteq I \times E$ and $(x, e) \in \phi$, if, and only if, $A(x)$ is isomorphic to $B(e)$. Therefore, ϕ is the graph of a one-to-one map of I onto E. It will now be shown that C is projective and so that ψ is projective.

Let $M = \{(x, \sigma) : \forall n, s_n \prec x\}$. We have $M = (I \times I^\omega) - \cup K_n$ where
$$K_n = \{(x, \sigma) : s_n \succcurlyeq x\}.$$

But, clearly the sets K_n are all projective sets of bounded class; M is a projective set.

Let $L = \{(x, \sigma) : [(y \prec x) \to \exists\, n \text{ such that } s_n = y]\}$.

Then

$$L = (I \times I^\omega) - Q,$$

where $Q = \{(x, 0) : \exists y \text{ with } (y \prec x) \text{ with } \forall n, s_n \neq y\}$.

Now, Q is the projection of J, where

$$J = \{(x, \sigma, y) : (y \prec x) \text{ and } \forall n, s_n \neq y\}$$

and $J = J_1 \cap J_2$, where

$$J_1 = \{(x, \sigma, y) : y \prec x\},$$

and

$$J_2 = \{(x, \sigma, y) : \forall n, s_n \neq y\}.$$

Clearly, J_1 is projective and J_2 is a G_δ-set.

Thus,

$$M \cap L = T_1 = \{(x, \sigma) : \forall n, s_n \prec x \text{ and } [(y \prec x) \to \exists n \text{ with } s_n = y]\}$$

is a projective subset of $I \times I^\omega$.

Let

$$T_2 = \{(\tau, e) : e \in E \text{ and } \tau \text{ is an enumeration of } B(e)\}.$$

It can be shown that T_2 is a projective subset of $E \times E^\omega$, in the same manner that T_1 was shown to be a projective subset of $I \times I^\omega$. Since $E \times E^\omega$ is a projective subset of $I \times I^\omega$ [1, p. 454], it follows that T_2 is a projective set.

Thus, $C = T_1 \times T_2$ is a projective set.

In order to show that there are $2^{\aleph_0}$ projective sets no two of which are Borel isomorphic, the following three lemmas are developed.

LEMMA 2. *Suppose A is an analytic subset of I^3. Let*

$$U = \{(x, y) : |z : (x, y, z) \in A| = 1\}.$$

Then U is a CPCA *subset of I^2.*

Proof. Let g, h, and k be Borel maps of I into I such that the map $\mu(t) = (g(t), h(t), k(t))$, $\forall t \in I$, takes I onto A.

Note $(x, y) \in I^2 - U$, if, and only if, (x, y) is not in the 12-projection of A or $\exists(\xi, \eta) \in I^2$ such that $x = g(\xi) = g(\eta)$, $y = h(\xi) = h(\eta)$, and $k(\xi) \neq k(\eta)$.

Let $Q_1 = [I^2 - (g, h)(I)] \times I^2$, where $(g, h)(I) = \{(g(t), h(t)) : t \in I\}$; Q_1 is a coanalytic subset of I^4. Let $Q_2 = \{(x, y, \xi, \eta) : x = g(\xi) = g(\eta) \text{ and } y = h(\xi) = h(\eta)\}$; Q_2 is an analytic subset of I^4. Let $Q_3 = \{(x, y, \xi, \eta) : k(\xi) \neq k(\eta)\}$; Q_3 is a Borel subset of I^4. It follows that $U = I^2 - \prod_{12}(Q_1 \cup (Q_2 \cap Q_3))$.

LEMMA 3. *There is a projective subset R of $I \times I^2$ such that,* (1) *each* 1-*section of R is the graph of a* 1-1 *projective function defined on an uncountable set, and* (2) *the* *graph of every* 1-1 *Borel function defined on an uncountable Borel subset of I is a* 1-*section of R.*

Proof. Let U be an analytic subset of $I \times I^2$ such that every Borel subset of I^2 is a 1-section of U.

Let $K_1 = \{(x_0, y_0)$: the line $x = x_0$, $y = y_0$ cuts U in exactly one point$\}$. Let $K_2 = \{(x_0, z_0)$: the line $x = x_0$, $z = z_0$ cuts U in exactly one point$\}$. K_1 and K_2 are CPCA subsets of I^2, by Lemma 2.

Let $G = \{x_0$: the line $x = x_0$ cuts K_1 in uncountably many points$\}$. Sierpiński has shown that G is a projective set [3, p. 63].

Let $R = U \cap (K_1 \times I) \cap (G \times I^2) \cap M_{(2,3)}(K_2 \times I)$. The set R has the required properties.

From here on for each $t \in I$, ϕ_t denotes the map whose graph is $R \cap (\{t\} \times I^2)$.

Recall that if $\mathscr{F}$ is a family of subsets of I and R is a relation (subset) of $\mathscr{F} \times I^n$, then R is said to be projective provided there is a map ψ of I onto $\mathscr{F}$ such that

$$E = \{(t, x_1, \ldots, x_n) : (\psi(t), x_1, \ldots, x_n) \in R\}$$

is projective [2, p. 137].

Lemma 4. *For each $t \in I$ and each subset X of I with $|X| \leqslant \aleph_0$, let*

$$F(X, t) = I - \left[\bigcup_{(z \prec t)} \phi_z(X) \cup X\right].$$

The propositional function $y \in F(X, t)$ is projective.

Proof. Let ψ be a map from I onto the space of all countable subsets of I such that $\psi = \{(t, x) : x \in \psi(t)\}$ is a projective subset of $I \times I$. In order to show that the propositional function $y \in F(X, t)$ is projective, it will be shown that the set

$$H = \{(a, t, y) : y \in F(\psi(a), t)\}$$

is projective. Note

$$(a, t, y) \notin H \Leftrightarrow \exists z[(z \preccurlyeq t) \wedge y \in \phi_z(\psi(a))] \vee y \notin \psi(a).$$

Let

$$Q_1 = \{(a, t, z, y) : y \notin \psi(a)\} = M_{(1,2)(3,4)}(I \times (I^2 - \psi) \times I).$$

Q_1 is a projective subset of I^4. Let $Q_2 = \{(a, t, z, y) : (z \preccurlyeq t)\}$; Q_2 is a projective subset of I^4. Let $Q_3 = \{(a, t, z, y) : y \in \phi_z(\psi(a))\}$. Let

$$K = [I \times M_{(2,3)}(R)] \cap M_{(1,2)(3,4)}(I \times \psi \times I),$$

where R is the set constructed in Lemma 3. Then $Q_3 = M_{(1,2)}(I \times \prod_{123}(K))$.

Thus,

$$I^3 - H = \textstyle\prod_{124} (Q_1 \cap (Q_2 \cap Q_3))$$

and therefore H is projective.

Theorem 2. *There are $2^{\aleph_0}$ projective subsets of the unit interval no two of which are Borel isomorphic.*

Proof. There is a unique function f from I into I such that

$$f(t) = p\{F[f(A(t)), t]\},$$

for every $t \in I$ [2, *p.* 137].

Let $x_\alpha = f(t_\alpha)$, where $\{t_\alpha\}_{\alpha<\omega_1}$ is the given projective well-ordering of the unit interval.

If $t_1, t_2 \in I$ and $t_1 \prec t_2$, then $x_1 = f(t_1) \in f\big(A(t_2)\big)$. Thus, $x_1 \notin F\Big(f\big(A(t_1)\big), t_2\Big)$ which means $x_1 \neq x_2 = f(t_2)$. So, f is one-to-one and $|P| = 2^{\aleph_0}$, where $P = f(I)$.

Also, if $t_1 \prec t_2$, then $\phi_{t_2}(x_1) \in \bigcup_{(t\prec t_2)} \phi_t\Big(f\big(A(t_2)\big)\Big)$ and therefore $x_2 \neq \phi_t(x_1)$.

Set $\phi_\eta = \phi_{t_\eta}$ for $\eta \prec \omega_1$. It has been shown that $x_\alpha \neq \phi_\eta(x_\beta)$, if $\beta \prec \alpha$ and $\eta \prec \alpha$ and $x_\eta \neq x_\alpha$, if $\eta \prec \alpha$.

Finally, the set P is projective since the propositional function $y \in F(X, t)$ is projective [2, p. 137].

It follows from Theorem 1 that there is a family $\mathscr{F}$ of $2^{\aleph_0}$ disjoint projective sets each of cardinality $2^{\aleph_0}$ which fills up P.

Now, suppose X and Y are sets in $\mathscr{F}$ and X is Borel isomorphic to Y. Let f be a Borel isomorphism of X onto Y as defined in the introduction. According to a theorem of Kuratowski's there are Borel sets Z_1 and Z_2 and a Borel isomorphism g of Z_1 onto Z_2 such that $Z_1 \supseteq X$, $Z_2 \supseteq Y$ and $g|X = f$ [1, *p*. 436]. There are ordinals γ, ξ with $\gamma, \xi < \omega_1$ such that $\phi_\gamma = g$ and $\phi_\xi = g^{-1}$.

The remainder of the argument follows exactly one given by Kuratowski [1, *p*. 426]:

Let α be an ordinal, $\xi \prec \alpha$ such that $x_\alpha = f(t_\alpha) \in X$. There is an ordinal η such that $\phi_\gamma(x_\alpha) = x_\eta = f(t_\eta)$. Therefore, $t_\alpha \preccurlyeq t_\eta$. Since $x_\eta \in V$, and $x_\alpha \notin y$, $t_\alpha \neq t_\eta$.

So, $t_\alpha \prec t_\eta$ and $x_\eta = \phi_\gamma(x_\alpha)$. Therefore, $t_\eta \prec t_\gamma$. But, this implies that Y is countable. This contradiction proves the theorem.

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