Random Linear Cellular Automata: Fractals associated with random multiplication of polynomials

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Abstract Random multiplication of a given set of s polynomials with coefficients in a finite field following a random sequence generated by Bernoulli trial with s possible outcomes is a (time-dependent) linear cellular automaton (LCA). As in the case of LCA with states in a finite field we associate with this sequence a compact set - the rescaled evolution set. The law of the iterated logarithm implies that this fractal set almost surely does not depend on the random sequence.

Keywords: random multiplication, polynomials, linear cellular automata, fractals

1 Introduction

It has been observed that the evolution patterns of seeds (initial configurations) with respect to many cellular automata exhibit self-similarity properties, [19], [24]. This phenomenon is especially apparent for the evolution patterns of finite seeds with respect to the additive (linear) cellular automata with states in the residue classes of the integers. Moreover, the self-reproducing property, the main idea of the inventors of cellular automata, J. v. Neuman and S. Ulam, includes a kind of self-similarity property, [2].

For the mathematical understanding of the problem of deciphering the self-similarity structure of the pattern evolution (orbit) of a finite seed with respect to linear cellular automata (LCA) with states in the residue classes of the integers modulo a prime number (or a prime power), the idea of rescaling proposed by S. Willson is important. In a series of papers, S. Willson associated with such an automata a compact set - the so called rescaled evolution set, [20], [22]. This set is a fractal and its self-similarity structure codes the self-similarity properties of the evolution patterns of the LCA. The self-similarity structure of the evolution set of LCA was described as a special graph directed construction (in the sense of Mauldin-Williams, [15]), in [6], [5], [7], and [18].

The idea of a rescaled evolution set was used in [13] for the description of the self-similarity properties of some classical number sequences - Gaussian binomial coefficients and Stirling numbers of the first and second kind modulo a prime power. A generalization of the notion of a linear cellular automaton necessary for this purpose, is a periodic time dependent cellular automaton. Such a cellular automaton is generated by a periodic multiplication of a finite number of polynomials.

The purpose of this note is the randomization of the notion of a periodic time dependent cellular automaton generated by s polynomials with coefficients in a Galois field and its geometrical

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representation - the rescaled evolution set. For every point of the code or shift space on s symbols, by appropriately rescaling its geometrical representation, we obtain a sequence of compact sets. One main question is the convergence of this sequence with respect to the Hausdorff distance. In general this sequence of compact sets does not converge, and even if it does converge, then its limit depends on the corresponding point of the shift space.

The main result (Theorem 1) in Section 7 is that a natural rescaled evolution set exists for almost all points of the code space (with respect to the Bernoulli measure generated by a given probability vector). Moreover, this "expected" rescaled evolution set does not depend on the choice of the point in the code space.

For the proof of this result we consider first in Sections 3, a deterministic situation which includes the "expected " rescaled evolution set and some of its specializations (for one polynomial, in Section 5, and in Section 4, for several polynomials and a rational probability vector or parameters). In the last cases the rescaled evolution sets are affinely equivalent with the rescaled evolution sets of appropriate linear cellular automata. In Section 8, we use this to calculate the Hausdorff dimension of the expected evolution set in some cases.

To make the notations more transparent we shall work with generating polynomials of the corresponding LCA or time dependent LCA.

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2 Preliminaries and notations

Let $\mathbb{F}_q$ be the Galois field with q elements (q is a power of some prime number p) and let $\mathbb{F}_q[x]$ be the ring of all polynomials with coefficients in $\mathbb{F}_q$. The most important and useful property of such polynomials $r \in \mathbb{F}_q[x]$ for this note is

$$(r(x))^q = r(x^q).$$

This was called the q-Fermat property in [7] since for q prime number it is equivalent with the small Fermat theorem, [10], pp. 70, 77.

Let $\mathcal{S} = (s_n)_n, s_n(x) = \sum_\alpha s(\alpha, n)x^\alpha \in \mathbb{F}_q[x]$. For the definition of the rescaled evolution set of the sequence $\mathcal{S}$ we need some notations. The set

$$X(\mathcal{S}) = \cup\{I(\alpha, n) | s(\alpha, n) \neq 0\},$$

where $I(\alpha, n) = [\alpha, \alpha + 1] \times [n, n + 1] \subset \mathbb{R}^2$, is called evolution set of the sequence $\mathcal{S}$.

The set $X(\mathcal{S})$ codes some basic information (zero or non-zero) about the coefficients of the polynomials $s_n \in \mathcal{S}$, arranged with respect to n (the "time").

Examples 1 1. *Linear cellular automaton $L(r)$ corresponding to a nonzero polynomial* $r \in \mathbb{F}_q[x]$.

The sequence of polynomials $\mathcal{S}(r) := (r^n)_{n\geq 0}$ is the orbit of the initial seed (configuration) $\delta = (\delta_{(0,n)})_{\mathbb{Z}}$ with respect to $L(r)$. Then $X(\mathcal{S})$ is a geometrical realization of this orbit in $\mathbb{R}^2$, [6], [7], [8]. This is the set we see on the computer screen visualizing the evolution of the initial seed δ under $L(r)$, [24]. Usually we shall use the notation $X(r)$ instead of $X(\mathcal{S})$.

2. Let $r_1, \ldots, r_s$ be nonzero polynomials with coefficients in $\mathbb{F}_q$ and $\mathbf{r} = (r_1, \ldots, r_s)$. Let $\theta_1, \ldots, \theta_s$ be a positive real numbers and $\theta = (\theta_1, \ldots, \theta_s)$. The sequence of polynomials $\mathcal{S}(\mathbf{r}; \theta) := (R_n)_{n\geq 0}$, where

$$R_n = \prod_{i=1}^{s} r_i^{[n\theta_i]}$$

is associated with the polynomials $r_1, \dots, r_s$ and parameters $\theta_1, \dots, \theta_s$. Here and later on we denote by $[a]$ the integer part of the real number a.

We shall denote the evolution set of the sequence $\mathcal{S}(\mathbf{r};\theta)$ by $X(\mathbf{r};\theta)$.

3. We shall consider also the sequence of polynomials $\mathcal{S}(\mathbf{r};\theta;\omega) := (R_{n,\omega})_{n\geq 0}$ associated with the polynomials

$$R_{n,\omega} = \prod_{i=1}^{s} r_i^{\omega^i(n)}$$

and the point $\omega \in \Omega = \{1, \dots, s\}^{\mathbb{N}}$, where $\omega^i(n) = card\{k \mid \omega(k) = i,\ k \leq n\}$.

The sequence $\mathcal{S}(\mathbf{r};\theta;\omega)$ for fixed ω is generated by the multiplication of the polynomials $r_1, \dots, r_s$ according to the point ω: $R_{n,\omega} = \prod_{i=1}^{s} r_{\omega(i)}$. We shall denote the evolution set of this sequence by $X(\mathbf{r};\theta;\omega)$.

For $m \in \mathbb{N}$, $m \geq 2$, the set

$$X_m(\mathcal{S}) = X(\mathcal{S}) \cap (\mathbb{R} \times [0, m])$$

is compact and nonempty in case at least one of the polynomials s_n of the sequence $\mathcal{S}$ is not zero. Then it is a point in the space $\mathcal{K}(\mathbb{R}^2)$ of all nonempty compact subsets of the plane $\mathbb{R}^2$. In this space we consider the Hausdorff metric ρ_H, generated by the l_∞-norm of $\mathbb{R}^2$, [11], pp. 214-215.

It is defined as follows : for a given $A \in \mathcal{K}(\mathbb{R}^2)$ and a positive number ϵ, let

$$A_\epsilon = \{x \mid ||x - y||_\infty < \epsilon, \text{ for some point } y \in A\},$$

where $||x||_\infty = \max\{|x_1|, |x_2|\}, x = (x_1, x_2) \in \mathbb{R}^2$.

Then the Hausdorff distance $\rho_H(A, B)$ between two sets $A, B \in \mathcal{K}(\mathbb{R}^2)$ is defined by

$$\rho_H(A, B) = \inf\{\epsilon | A \subset B_\epsilon, B \subset A_\epsilon\}.$$

The metric space $(\mathcal{K}(\mathbb{R}^2), \rho_H)$ is complete, moreover, if $K \in \mathcal{K}(\mathbb{R}^2)$ the space $(\mathcal{K}(K), \rho_H)$ of all nonempty compact subsets of K is compact, [11], pp. 216, 407.

The sets $X_m(\mathcal{S})$ are compact but their union is unbounded in $\mathbb{R}^2$. For this reason they are rescaled by appropriate similitudes.

The increasing sequences $\underline{a} = (a(n))_{n\geq 0}, \underline{b} = (b(n))_{n\geq 0} \in \mathbb{N}^{\mathbb{N}}$ such that

$$\max\{\frac{B(n)}{a(n)}, \frac{b(n)}{a(n)}\} \leq C$$

where $d_k = \deg s_k, B(n) = \max\{d_k \mid k \leq b(n) - 1\}$ and C is an appropriate constant are called scaling sequences if the sequence of compact sets

$$(s_{a(n)^{-1}}(X_{b(n)}(\mathcal{S})))_{n\geq 0}$$

converges with respect to the Hausdorff metric ρ_H.

Here $s_c : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ is the similitude $s_c(x, y) = (cx, cy), (x, y) \in \mathbb{R}^2$.

Observe that

$$s_{a(n)^{-1}}(X_{b(n)}(\mathcal{S})) \subset [0,C] \times [0,C].$$

The limit

$$\lim_{n\to\infty} s_{a(n)^{-1}}(X_{b(n)}(\mathcal{S}))$$

is called rescaled evolution set of $\mathcal{S}$ (with respect to the scaling sequences $\underline{a}, \underline{b} \in \mathbb{N}^{\mathbb{N}}$).

Examples 2 The sequences $\underline{a} = \underline{b} = (q^t)_{t\geq 0}$ are scaling sequences for the sequence $X(r)$ for every nonzero polynomial r with coefficients in $\mathbb{F}_q$, [20]. The rescaled evolution set corresponding to these sequences is denoted by $A_q(r)$. In general this is a fractal set - for example for $r(x) = 1 + x \in \mathbb{F}_2[x]$ the set $A_2(r)$ is the Sierpinski triangle. For more examples see [6], [5], [7], [8].

In this note we shall consider the scaling sequences $\underline{a} = (q^t)_{t\geq 0}$ and $\underline{b} = ([aq^t])_{t\geq 0}$ for a positive number a. We shall call them standard scaling sequences. In the last section we consider also some other scaling sequences.

3 Existence of rescaled evolution set for several polynomials - general case

Here we consider the nonzero polynomials $r_1, \ldots, r_s$ with coefficients in the Galois field $\mathbb{F}_q$ and positive real numbers (parameters) $a, \theta_1, \ldots, \theta_s$. The evolution set of the sequence of polynomials $\mathcal{S}(\mathbf{r};\theta) = (R_n)_{n\geq 0}$, where

$$R_n(x) = \prod_1^s (r_i(x))^{[n\theta_i]} = \sum_\alpha R(\alpha, n) x^\alpha$$

is the set

$$X(\mathbf{r};\theta) = \bigcup \{I(\alpha, n) | R(\alpha, n) \neq 0\},$$

where $\mathbf{r} = (r_1, \ldots, r_s)$ and $\theta = (\theta_1, \ldots, \theta_s)$.

The compact sets to be rescaled are

$$X_{[aq^t]}(\mathbf{r};\theta) = X(\mathbf{r},\theta) \bigcap (\mathbb{R} \times [0, [aq^t]]).$$

The aim of this section is the following

Proposition 1 *The sequence of compact sets*

$$(s_{q^{-t}}(X_{[aq^t]}(\mathbf{r};\theta))_{t\geq 0} \tag{1}$$

converges with respect to the Hausdorff metric ρ_H.

Proof

The metric space $(\mathcal{K}(\mathbb{R}^2), \rho_H)$ is complete. Therefore the assertion is that the sequence (1) is a Cauchy sequence with respect to the Hausdorff metric ρ_H. It suffices to prove that there exists a constant C such that

$$\rho_H(s_{q^{-1}}(X_{[aq^{t+1}]}(\mathbf{r};\theta)), X_{[aq^t]}(\mathbf{r};\theta)) \leq C \tag{2}$$

for all $t \in \mathbb{N}$.

The estimation (2) follows from the inclusions

$$s_{q^{-1}}(X_{[aq^{t+1}]}(\mathbf{r};\theta)) \subset (X_{[aq^t]}(\mathbf{r};\theta))_C, \tag{3}$$

$$X_{[aq^t]}(\mathbf{r};\theta) \subset (s_{q^{-1}}(X_{[aq^{t+1}]}(\mathbf{r};\theta)))_C \tag{4}$$

for all $t \in \mathbb{N}$, where as defined earlier the subscript C indicates all points whose distance to the set is less than C.

Proof of (3)

Let

$$I(k,n) \subset X_{[aq^{t+1}]}(\mathbf{r};\theta), \tag{5}$$

i.e., the coefficient $R(k,n)$ of the polynomial R_n - the n-th element of the sequence $\mathcal{S}(\mathbf{r};\theta)$ - is not zero and $n < [aq^{t+1}]$.

We shall prove first that the polynomial

$$\hat{R}_n(x) = \prod_1^s (r_i(x))^{[[\frac{n}{q}]\theta_i]q} = \sum_\alpha \hat{R}(\alpha,n)x^\alpha \tag{6}$$

is a factor of the polynomial R_n, i. e.,

$$R_n = \hat{R}_n \tilde{R}_n \tag{7}$$

for some polynomial $\tilde{R}_n \in \mathbb{F}_q[x]$.

This follows from the inequalities

$$0 \le [n\theta_i] - [[\frac{n}{q}]\theta_i]q \le q(\Theta+1), i = 1,\dots,s, \tag{8}$$

where $\Theta = \max\{\theta_i | i = 1,\dots,s\}$.

In fact, let $n = lq + j$ where $l \in \mathbb{N}$ and $0 \le j \le q-1$. Then

$$lq\theta_i - q \le [[\frac{n}{q}]\theta_i]q \le [n\theta_i] < lq\theta_i + q\Theta,$$

which implies (8).

The inequality (8) and (7) imply also that

$$\deg \tilde{R}_n \le Dq(\Theta+1), \tag{9}$$

where $D = \sum_1^s d_i$ and $d_i = \deg r_i$, $i = 1,\dots,s$.

The coefficient $R(k,n)$ of the polynomial R_n is not zero. Then from (7) and (9) follows: there exists $k_1 \in \mathbb{N}$ such that the k_1-th coefficient $\hat{R}(k_1,n)$ of the polynomial $\hat{R}_n$ satisfies

$$\hat{R}(k_1,n) \ne 0 \tag{10}$$

$$k - Dq(\Theta+1) \le k_1 \le k. \tag{11}$$

Consider the polynomial

$$R_{[\frac{n}{q}]}(x) = \prod_1^s (r_i(x))^{[[\frac{n}{q}]\theta_i]} = \sum_\alpha R(\alpha, [\frac{n}{q}])x^\alpha.$$

This is the $[\frac{n}{q}]$-th element of the sequence $\mathcal{S}(\mathbf{r};\theta)$.

From (6) follows

$$\hat{R}_n(x) = (R_{[\frac{n}{q}]}(x))^q = \sum_\alpha R(\alpha, [\frac{n}{q}])x^{\alpha q}. \tag{12}$$

Then (10) and (12) imply that

$$k_1 = k_2 q, \tag{13}$$

and

$$R(k_2, [\frac{n}{q}]) \neq 0,$$

which gives

$$I(k_2, [\frac{n}{q}]) \subset X_{[aq^t]}(\mathbf{r};\theta), \tag{14}$$

(here we used $[\frac{n}{q}] \leq \frac{n}{q} < aq^t$).

From (11), (13) and (14) follows

$$s_{q^{-1}}(I(k,n)) \subset (I(k_2, [\frac{n}{q}]))_{C_1} \subset (X_{[aq^t]}(\mathbf{r};\theta))_{C_1}, \text{ for } C_1 = D(\Theta+1),$$

and therefore

$$s_{q^{-1}}(X_{[aq^{t+1}]}(\mathbf{r};\theta)) \subset (X_{[aq^t]}(\mathbf{r};\theta))_{C_1}. \tag{15}$$

Proof of (4)

Let

$$I(k,n) \subset X_{[aq^t]}(\mathbf{r};\theta),$$

i.e., the coefficient $R(k,n)$ of the polynomial R_n - the n-th element of the sequence $\mathcal{S}(\mathbf{r};\theta)$ - is not zero and $n < [aq^t]$.

Consider the polynomial

$$(R_n(x))^q = \prod_1^s (r_i(x))^{[n\theta_i]q} = \sum_\alpha R(\alpha, n)x^{\alpha q}, \tag{16}$$

(here we are using the q-Fermat property).

Case $n \geq \frac{1}{\theta} + 1$

The polynomial

$$R_{n_1} = \prod_1^s r_i^{[n_1\theta_i]}$$

is a factor of the polynomial $(R_n)^q$ for

$$n_1 = nq - [\frac{q}{\theta}] - 1,$$

where

$$\theta = \min\{\theta_i | i = 1, \ldots, s\}, \text{ and } n \geq \frac{1}{\theta} + 1,$$

i.e.,

$$(R_n)^q = R_{n_1} \tilde{R}_{n_1}, \tag{17}$$

and $\tilde{R}_{n_1}$ is a polynomial with coefficients in $\mathbb{F}_q$. Moreover, R_{n_1} is the n_1-th polynomial of the sequence $\mathcal{S}(\mathbf{r};\theta)$.

In fact $n_1 \geq 0$ and

$$[nq\theta_i] - \Theta([\frac{q}{\theta}] + 1) - 1 \leq [n_1\theta_i] \leq q[n\theta_i] \leq [nq\theta_i], \tag{18}$$

since

$$[n_1\theta_i] \leq n_1\theta_i \leq nq\theta_i - q\frac{\theta_i}{\theta} \leq q(n\theta_i - 1) \leq q[n\theta_i] \leq [nq\theta_i],$$

and

$$[n_1\theta_i] \geq n_1\theta_i - 1 = nq\theta_i - \theta_i([\frac{q}{\theta}] + 1) - 1 \geq [nq\theta_i] - \Theta([\frac{q}{\theta}] + 1) - 1.$$

Moreover,

$$0 \leq n - \frac{n_1}{q} \leq \frac{1}{q}([\frac{q}{\theta}] + 1) \leq \frac{1}{\theta} + 1. \tag{19}$$

The coefficient $R(k,n)$ of the polynomial R_n is not zero, then from (16) and (17) follows that there exists $k_1 \in \mathbb{N}$ such that the k_1-th coefficient $R(k_1,n_1)$ of the polynomial R_{n_1} is not zero and

$$kq - \deg \hat{R}_{n_1} \leq k_1 \leq kq. \tag{20}$$

Then (18) implies

$$\deg \hat{R}_{n_1} \leq D\{\Theta([\frac{q}{\theta}] + 1) + 1\}. \tag{21}$$

Therefore

$$I(k_1,n_1) \subset X_{[aq^{t+1}]}(\mathbf{r};\theta). \tag{22}$$

From (19) , (20), (21), and (22) follows

$$I(k,n) \subset (s_{q^{-1}}(I(k_1,n_1)))_{C_2} \subset (s_{q^{-1}}(X_{[aq^{t+1}]}(\mathbf{r};\theta)))_{C_2} \tag{23}$$

for $n \geq \frac{1}{\theta} + 1$ and $C_2 = \max\{\frac{1}{\theta} + 1, \frac{D}{q}\{\Theta([\frac{q}{\theta}] + 1) + 1\}$.

<u>*Case* $n < \frac{1}{\theta} + 1$</u>

The inclusion

$$I(k,n) \subset (s_{q^{-1}}(X_{[aq^{t+1}]}(\mathbf{r};\theta)))_{\frac{D\Theta}{\theta}(\frac{1}{\theta}+1)}, \tag{24}$$

follows since

$$0 \le k \le \sum_1^s d_i[n\theta_i] \le \sum_1^s nd_i\frac{\theta_i}{\theta} \le \frac{D\Theta}{\theta}(\frac{1}{\theta}+1),$$

and

$$\deg R_{nq} \le \sum_1^s d_i[nq\theta_i] \le q\frac{D\Theta}{\theta}(\frac{1}{\theta}+1),$$

and there exists l with $0 \le l \le q\frac{D\Theta}{\theta}(\frac{1}{\theta}+1)$ with $I(l,nq) \subset X_{[aq^{t+1}]}(\mathbf{r};\theta)$. Then $|k-\frac{l}{q}| \le \frac{D\Theta}{\theta}(\frac{1}{\theta}+1)$.

Combining (23) and (24) we obtain

$$X_{[aq^t]}(\mathbf{r};\theta) \subset (s_{q^{-1}}(X_{[aq^{t+1}]}(\mathbf{r};\theta)))_{C_3}, \tag{25}$$

where $C_3 = \max\{C_2, \frac{D\Theta}{\theta}(\frac{1}{\theta}+1)\}$.

From (25) and (15) follows (2) with $C = \max\{C_1, C_3\}$.

□

We denote by $A_q(\mathbf{r};\theta;a)$ the limit of the sequence (1) with respect to the Hausdorff metric ρ_H:

$$A_q(\mathbf{r};\theta;a) = \lim_{t\to\infty} s_{q^{-t}}(X_{[aq^t]}(\mathbf{r};\theta)).$$

Remark 1 For $s = \theta_1 = a = 1, r = r_1$ the set $A_q(r;1;1)$ is the rescaled evolution set $A_q(r)$ associated with polynomial r (or with the linear cellular automaton generated by the polynomial r), [20], [7], [8].

For different values of the parameter a the sets $A_q(\mathbf{r};\theta;a)$ are different and in some cases nonhomeomorphic. For example let $q = 2, s = 1, r(x) = r_1(x) = 1 + x \in \mathbb{F}_2[x], \theta_1 = 1$. The sets $A_2(r;1;3)$ and $A_2(r)$ are not homeomorphic. The second set is the Sierpinski triangle and the first is given by

$$A_2(r;1;3) = s_4(A_2(r)) \cap ([0,3] \times [0,3]). \tag{26}$$

The sets $A_2(r;1;3)$ and $A_2(r)$ are not homeomorphic, since the Sierpinski triangle has only 3 points - $(0,0),(1,1),(1,0)$ - with branching index 2 and from (26) follows that the set $A_2(r;1;3)$ has 5 points with branching index 2 - $(0,0),(0,3),(1,3),(2,3)$ and $(3,3)$, [16], pp. 125-126.

4 Rescaled evolution set for several polynomials and rational parameters θ_i

Here we shall specialize the situation from the previous section. We consider the nonzero polynomials $r_1,\ldots,r_s \in \mathbb{F}_q[x]$, a positive real number a and positive rational numbers $\theta_1,\ldots,\theta_s$. Assume that $\theta_i = \frac{k_i}{l}$, where l and $k_i, i = 1,\ldots,s$ are positive integers and $a = l$. By $F_l : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ we denote the affine map defined with $F_l(x,y) = (x, \frac{y}{l})$ for $(x,y) \in \mathbb{R}^2$.

From Proposition 1 we know that the sequence $(s_{q^{-t}}(X_{lq^t}(\mathbf{r};\theta)))_{t\geq 0}$ converges to the set $A_q(\mathbf{r};\theta;l)$ with respect to the Hausdorff metric ρ_H. Here we shall describe this set. For this description we shall use the representation of the rational numbers $\theta_i = \frac{k_i}{l}$ and shall write $X_{lq^t}(\mathbf{r};(\frac{k_1}{l},\ldots,\frac{k_s}{l}))$ instead of $X_{lq^t}(\mathbf{r};\theta)$. What we show is that the limit set associated with this vector of polynomials and rational parameter values is the affine image of the limit set associated with a single polynomial.

Proposition 2 *The sequence of compact sets $(s_{q^{-t}}(X_{lq^t}(\mathbf{r};(\frac{k_1}{l},\ldots,\frac{k_s}{l}))))_{t\geq 0}$ converges to the set $F_l^{-1}(A_q(r_1^{k_1}\cdots r_s^{k_s}))$.*

Proof

The assertion follows from the inequality

$$\rho_H(F_l(X_{lq^t}(\mathbf{r};(\frac{k_1}{l},\ldots,\frac{k_s}{l}))), X_{q^t}(r_1^{k_1}\cdots r_s^{k_s};1)) \leq \Delta \tag{27}$$

for all $t \in \mathbb{N}$, where $\Delta = \sum_1^s d_i k_i$ and $d_i = \deg r_i$.

The inequality (27) follows from the following two inclusions

$$F_l(X_{lq^t}(\mathbf{r},(\frac{k_1}{l},\ldots,\frac{k_s}{l}))) \subset (X_{q^t}(r_1^{k_1}\cdots r_s^{k_s};1))_\Delta \tag{28}$$

$$X_{q^t}(r_1^{k_1}\cdots r_s^{k_s};1) \subset (F_l(X_{lq^t}(\mathbf{r},(\frac{k_1}{l},\ldots,\frac{k_s}{l}))))_\Delta \tag{29}$$

Proof of (28)

Let

$$I(a,n) \subset X_{lq^t}(\mathbf{r};(\frac{k_1}{l},\ldots,\frac{k_s}{l})),$$

i.e., the coefficient $R(a,n)$ of the polynomial R_n - the n-th element of the sequence $\mathcal{S}(\mathbf{r};(\frac{k_1}{l},\ldots,\frac{k_s}{l}))$ is not zero and $n < lq^t$.

Let $n = lm + j, 0 \leq j \leq l-1, m \in \mathbb{N}$ and let $jk_i = u_i l + v_i, 0 \leq v_i \leq l-1, u_i \in \mathbb{N}, i = 1,\ldots,s$. Then

$$[n\theta_i] = [n\frac{k_i}{l}] = mk_i + u_i,\ i = 1,\ldots,s$$

and

$$R_n = \prod_1^s r_i^{[n\theta_i]} = (r_1^{k_i}\cdots r_s^{k_s})^m r_1^{u_1}\cdots r_s^{u_s}. \tag{30}$$

Therefore the polynomial $R_{ml}(x) = (r_1(x)^{k_i}\cdots r_s(x)^{k_s})^m = \sum_\alpha R(\alpha,ml)x^\alpha$ is a factor of the polynomial R_n. Since the coefficient $R(a,n)$ of the polynomial R_n is not zero, then there is an integer a_1 such that

$$R(a_1,ml) \neq 0 \tag{31}$$

$$a - \Delta \leq a - \deg r_1^{u_1}\cdots r_s^{u_s} \leq a_1 \leq a. \tag{32}$$

Here R_{ml} is the m-th polynomial of the sequence $\mathcal{S}(r^{k_1}\cdots r^{k_s};1)$
From (31) and (32) follow

$$I(a_1,m)\subset X_{q^t}(r_1^{k_i}\cdots r_s^{k_s};1)$$
$$F_l(I(a,n))\subset (I(a_1,m))_\Delta.$$

The last two inclusions imply (28).

Proof of (29)

Assume that

$$I(c,m)\subset X_{q^t}(r_1^{k_1}\cdots r_s^{k_s};1),$$

i.e., the coefficient $R(c,ml)$ of the polynomial $R_{ml}=(r_1^{k_i}\cdots r_s^{k_s})^m$ - the m-th element of the sequence $\mathcal{S}(r^{k_1}\cdots r^{k_s};1)$ is not zero and $m<q^t$. This polynomial R_{ml} is also the ml-th element of the sequence $\mathcal{S}(\mathbf{r};(\frac{k_1}{l},\ldots,\frac{k_s}{l}))$. Therefore

$$I(c,lm)\subset X_{lq^t}(\mathbf{r};(\frac{k_1}{l},\ldots,\frac{k_s}{l}))$$

and

$$I(c,m)\subset (F_l(I(c,lm)))_1\subset (X_{lq^t}(\mathbf{r};(\frac{k_1}{l},\ldots,\frac{k_s}{l})))_1,$$

which implies (29) since $1\leq\Delta$.

□

Remark 2 The limit of the sequence $(s_{q^{-t}}(X_{lq^t}(\mathbf{r};(\frac{k_1}{l},\ldots,\frac{k_s}{l}))))_{n\geq 0}$ depends on the representation of the rational numbers θ_i as fractions:

$$F_{ul}^{-1}(A_q(r^{k_1u}\cdots r^{k_su}))=\lim_{t\to\infty}s_{q^{-t}}(X_{ulq^t}(\mathbf{r};(\frac{uk_1}{lu},\ldots,\frac{uk_s}{lu})),\text{ for }u\in\mathbb{N}.$$

and these sets are in general different.

5 Rescaled evolution set for one polynomial

Here we shall consider another specialization of the situation considered in the Section 3: $s=1,r=r_1,a=\frac{1}{\theta}$ and θ a positive real number. Let $F:\mathbb{R}^2\longrightarrow\mathbb{R}^2$ be the affine map defined with $F(x,y)=(x,\theta y)$ for $(x,y)\in\mathbb{R}^2$.

From Proposition 1 we know that the sequence $(s_{q^{-t}}(X_{[\frac{q^t}{\theta}]}(r;\theta)))_{t\geq 0}$ converges to the set $A_q(r;\theta;\frac{1}{\theta})$ with respect to the Hausdorff metric ρ_H. Here we shall describe this set.

Proposition 3 *The sequence of compact sets* $(s_{q^{-t}}(X_{[\frac{q^t}{\theta}]}(r;\theta)))_{t\geq 0}$ *converges to the set* $F^{-1}(A_q(r))$, *i.e.,* $A_q(r;\theta;\frac{1}{\theta})=F^{-1}(A_q(r))$.

Proof
The assertion follows from the estimation

$$\rho_H(F(X_{[\frac{q^t}{\theta}]}(r;\theta)), X_{q^t}(r)) \le C,$$

where $C = \max\{d(\theta+1), \theta+3\}$ and $d = \deg r$, which is equivalent with the inclusions

$$F(X_{[\frac{q^t}{\theta}]}(r;\theta)) \subset (X_{q^t}(r;1))_C, \tag{33}$$
$$X_{q^t}(r;1) \subset (F(X_{[\frac{q^t}{\theta}]}(r;\theta)))_C. \tag{34}$$

Proof of (33)

Let

$$I(k,n) \subset X_{[\frac{q^t}{\theta}]}(r;\theta),$$

i.e., the coefficient $R(k,n)$ of the polynomial

$$(r(x))^{[n\theta]} = \sum_\alpha R(\alpha,n)x^\alpha$$

is not zero. The polynomial $r^{[n\theta]}$ is the n-th element of the sequence $\mathcal{S}(r;\theta)$. It is also $[n\theta]$-th element of the sequence $\mathcal{S}(r;1)$. Therefore

$$I(k,[n\theta]) \subset X_{q^t}(r;1).$$

It follows that

$$F(I(k,n)) \subset (I(k,[n\theta]))_1.$$

The last inclusion implies (33).

Proof of (34)

Let

$$I(k,n) \subset X_{q^t}(r),$$

i.e., the coefficient $r(k,n)$ of the polynomial

$$(r(x))^n = \sum_\alpha r(\alpha,n)x^\alpha$$

is not zero (r^n is the n-th polynomial of the sequence $\mathcal{S}(r)$).
The polynomial $r^{[[\frac{n}{\theta}]\theta]}$ is a factor of the polynomial r^n since

$$n-\theta-1 \le [[\frac{n}{\theta}]\theta] \le n.$$

Since the coefficient $r(k,n)$ of the polynomial r^n is not zero, then there is a natural number k_1 such that the coefficient $R(k_1,[\frac{n}{\theta}])$ of the polynomial $r^{[[\frac{n}{\theta}]\theta]}$ satisfies

$$R(k_1,[\frac{n}{\theta}]) \neq 0, \tag{35}$$

$$k-d(\theta+1) \leq k_1 \leq k. \tag{36}$$

The polynomial $r^{[[\frac{n}{\theta}]\theta]}$ is the $[\frac{n}{\theta}]$-th element of the sequence $\mathcal{S}(r;\theta)$. Then from (35) follow

$$I(k_1,[\frac{n}{\theta}]) \subset X_{[\frac{q^t}{\theta}]}(r;\theta).$$

From (36) and the inequality

$$0 \leq n-[[\frac{n}{\theta}]\theta] \leq [\theta+2]$$

follows

$$I(k,n) \subset (F(I(k_1,[\frac{n}{\theta}])))_C,$$

which implies (34).

□

6 Rescaled evolution set for a perturbation of the sequence $\mathcal{S}(\mathbf{r};\theta)$

Here we shall consider some perturbations of the sequence of polynomials $\mathcal{S}(\mathbf{r};\theta)$ under which the rescaled evolution set is stable. Let $r_1,\ldots,r_s$ be nonzero polynomials with coefficients in $\mathbb{F}_q$ and let $a,\theta_i,\ldots,\theta_s$ be positive real numbers. We consider also the functions $h,g_i:\mathbb{N}\longrightarrow\mathbb{N}$ satisfying

- The function h is nondecreasing,
- $\lim_{n\to\infty}\frac{h(n)}{n}=0$,
- $|g_i(n)-[n\theta_i]|\leq h(n)$ for all $n\geq N$, where N is fixed natural number.

The function h is $o(n)$ for $n\to+\infty$ and the functions g_i are small (h-small) perturbation of the functions $\nu_i:\mathbb{N}\longrightarrow\mathbb{N}$, given by $\nu_i(n)=[n\theta_i]$, for $n\in\mathbb{N}$.

The condition that the function h is nondecreasing is not important but technically convenient. We shall choose the natural number N so big that

$$\frac{h(n)}{n} \leq \frac{\theta}{2(\Theta+2)},\ \text{and}\ h(n)\geq 1,\ \text{for}\, n\geq N, \tag{37}$$

where $\theta=\min\{\theta_i|i=1,\ldots,s\}$ and $\Theta=\max\{\theta_i|i=1,\ldots,s\}$.

Consider the polynomials

$$R_{n,\mathbf{g}}(x)=\prod_1^s(r_i(x))^{g_i(n)}=\sum_\alpha \tilde{R}(\alpha,n)x^\alpha$$

and the sequence $\mathcal{S}(\mathbf{r};\mathbf{g})=(R_{n,\mathbf{g}})_{n\geq 0}$, where $\mathbf{r}=(r_1,\ldots,r_s)$ and $\mathbf{g}=(g_1,\ldots,g_s)$. The sequence $\mathcal{S}(\mathbf{r};\mathbf{g})$ is a small perturbation of the sequence $\mathcal{S}(\mathbf{r};\theta)$.

The goal of this section is the following

Proposition 4

$$\lim_{t\to\infty} s_{q^{-t}}(X_{[aq^t]}(\mathbf{r};\mathbf{g})) = A_q(\mathbf{r};\theta;a),$$

i.e., the rescaled evolution set of the sequence $\mathcal{S}(\mathbf{r};\mathbf{g})$ does not depend on the small perturbation $\mathbf{g}$.

Proof
The assertion follows from the estimation

$$\rho_H(X_{[aq^t]}(\mathbf{r};\mathbf{g}), X_{[aq^t]}(\mathbf{r};\theta)) \le Ch([aq^t]),$$

for some constant C and for all sufficiently large $t \in \mathbb{N}$, since $\lim_{t\to\infty} \frac{h([aq^t])}{q^t} = 0$.
The last estimation is equivalent with the inclusions

$$X_{[aq^t]}(\mathbf{r};\mathbf{g}) \subset (X_{[aq^t]}(\mathbf{r};\theta))_{Ch([aq^t])}, \tag{38}$$
$$X_{[aq^t]}(\mathbf{r};\theta) \subset (X_{[aq^t]}(\mathbf{r};\mathbf{g}))_{Ch([aq^t])}. \tag{39}$$

Proof of (38), case $n \ge N$

Let $N \le n < [aq^t]$.
Define

$$m = [n - \frac{h(n)}{\theta}].$$

Then from (37) follows that m is a nonnegative integer. Moreover,

$$n - \frac{h(n)}{\theta} - 1 \le m \le n - \frac{h(n)}{\Theta},$$

which implies

$$0 \le n - m \le \frac{2}{\theta} h([aq^t]). \tag{40}$$

The polynomial R_m is a factor of the polynomial $R_{n,\mathbf{g}}$, since

$$n\theta_i - h(n)\frac{\Theta}{\theta} - \Theta \le [m\theta_i] \le g_i(n) \le n\theta_i + h(n). \tag{41}$$

Let

$$R_{n,\mathbf{g}} = R_m \hat{R}_m, \tag{42}$$

for some polynomial $\hat{R}_m$ with coefficients in $\mathbb{F}_q$. From (41) follows

$$\deg \hat{R}_m \le D(\frac{\Theta}{\theta} + \Theta + 1)h([aq^t]). \tag{43}$$

Assume that

$$I(k,n) \subset X_{[aq^t]}(\mathbf{r};\mathbf{g}),$$

i. e., the coefficient $\tilde{R}(k,n)$ of the polynomial $R_{n,\mathbf{g}}$ is not zero. Then (42) implies that there exists a natural number k_1 such that the k_1-th coefficient $R(k_1,n)$of the polynomial R_m is not zero, which means that $I(k_1,m) \subset X_{[aq^t]}(\mathbf{r};\theta)$. Moreover, (43) implies

$$k - D(\frac{\Theta}{\theta} + \Theta + 1)h([aq^t]) \le k_1 \le k \tag{44}$$

Then from (40) and (44) follows

$$I(k,n) \subset (I(k_1,m))_{C_1 h([aq^t])}, \tag{45}$$

where $C_1 = \max\{\frac{2}{\theta}, D(\frac{\Theta}{\theta} + \Theta + 1)\}$.

Proof of (38), case $n < N$

Assume

$$I(k,n) \subset X_{[aq^t]}(\mathbf{r};\mathbf{g}), \text{ and } n < N,$$

then

$$0 \le k \le DM, \tag{46}$$

where $D = \sum_1^s d_i, d_i = \deg r_i$, and $M = \max\{g_i(n) | n < N\}$.

Let $I(l,n) \subset X_{[aq^t]}(\mathbf{r};\theta)$. Since the degree of the n-th element R_n of the sequence $\mathcal{S}(\mathbf{r};\theta)$ is bounded from above by the number $DN\Theta$, then

$$0 \le l \le DN\Theta. \tag{47}$$

From (46) and (47) follows

$$I(k,n) \subset (I(l,n))_{C_2}, \tag{48}$$

where $C_2 = D\max\{N\Theta, M\}$, since $|k-l| \le C_2$.

From (45) and (48) follow (38) for $C \ge \max\{C_1, C_2\}$.

Proof of (39), case $n \ge 2N$

Let $0 \le 2N \le n < [aq^t]$.

Define

$$m = n - [\frac{2+\Theta}{\theta} h(n)].$$

From (37) follows that $m \ge N$.

Therefore

$$g_i(n) \le [m\theta_i] + h(m) \le n\theta_i - h(n)(\Theta+1) + \theta_i \le n\theta_i - 1 \le [n\theta_i] \le n\theta_i,$$

and

$$g_i(m) \ge n\theta_i - \{(2+\Theta)\frac{\Theta}{\theta} + 2\}h([aq^t]).$$

Then

$$0 \le [n\theta_i] - g_i(m) \le \{(2+\Theta)\frac{\Theta}{\theta} + 2\}h([aq^t]). \tag{49}$$

Therefore the polynomial $R_{m,\mathbf{g}}$ - the m-th polynomial of the sequence $\mathcal{S}(\mathbf{r};\mathbf{g})$ - is a factor of the polynomial R_n - the n-th polynomial of the sequence $\mathcal{S}(\mathbf{r};\theta)$:

$$R_n = R_{m,\mathbf{g}}\bar{R}_m, \tag{50}$$

and from (49)

$$\deg \bar{R}_m \le D\{(2+\Theta)\frac{\Theta}{\theta} + 2\}h([aq^t]). \tag{51}$$

Suppose $I(k,n) \subset X_{[aq^t]}(\mathbf{r};\theta)$ and $2N \le n \le [aq^t]$. Then (50) imply that there exists a natural number k_1 such that k_1 coefficient $\tilde{R}(k_1,m)$ of the polynomial $R_{m,\mathbf{g}}$ is not zero and moreover, from (51) follows

$$k - C_3 h([aq^t]) \le k_1 \le k,$$

for $C_3 = D\{(2+\Theta)\frac{\Theta}{\theta} + 2\}$. From the last inequality and

$$0 \le n - m \le \frac{2+\Theta}{\theta}h([aq^t])$$

follows

$$I(k,n) \subset (I(k_1,m))_{C_4 h([aq^t])}, \tag{52}$$

where $C_4 = \max\{C_3, \frac{2+\Theta}{\theta}\}$.

<u>*Proof of (39), case $n < 2N$*</u>

Since

$$\deg R_n < 2ND\Theta,$$

and

$$\deg R_{n,\mathbf{g}} \le M_1 D, \text{ where } M_1 = \max\{g_i(n) | n < 2N\},$$

then for $I(k,n) \subset X_{[aq^t]}(\mathbf{r};\theta)$ and $I(l,n) \subset X_{[aq^t]}(\mathbf{r};\mathbf{g})$ follows

$$I(l,n) \subset (I(k,n))_{C_5}, \tag{53}$$

where $C_5 = D\max\{M_1, 2N\Theta\}$.

Then (39) follows from (51) and (52) for $C \ge \max\{C_4, C_5\}$.

Therefore the proposition is proved for $C = \max\{C_1, C_2, C_4, C_5\}$.

□

7 Random multiplication of polynomials - the main theorem

The goal of this section is the proof of the theorem:

Theorem 1 *Let* $\mathbf{r} = (r_1, \ldots, r_s)$, $r_1, \ldots, r_s \in \mathbb{F}_q[x]$ *(nonzero polynomials) and let* $a, \theta = (\theta_1, \ldots, \theta_s), a > 0, \theta_i > 0, i = 1, \ldots, s, \sum_1^s \theta_i = 1$. *Then for almost all points* $\omega \in \Omega = \{1, \ldots, s\}^{\mathbb{N}}$ *with respect to the Bernoulli measure* μ_θ, *induced by the probability vector* θ *follows:*

a) The sequence of compact sets $(s_{q^{-t}}(X_{[aq^t]}(\mathbf{r};\theta;\omega)))_{t\geq 0}$ *converges with respect to the Hausdorff metric and*

$$A_q(\mathbf{r};\theta;a) = \lim_{t\to\infty} s_{q^{-t}}(X_{[aq^t]}(\mathbf{r};\theta;\omega)).$$

b) For $s = 2, r_1 = r, r_2 = 1, \theta_1 = \theta, \theta_2 = 1-\theta$ *the sequence of compact sets* $(s_{q^{-t}}(X_{[\frac{q^t}{\theta}]}(\mathbf{r};\theta;\omega)))_{t\geq 0}$ *converges with respect to he Hausdorff metric and*

$$A_q(r) = \lim_{t\to\infty} s_{q^{-t}}(X_{[\frac{q^t}{\theta}]}(\mathbf{r};\theta;\omega)).$$

c) For a rational probability vector $\theta = (\frac{k_1}{l}, \ldots, \frac{k_s}{l}), l, k_1, \ldots, k_s \in \mathbb{N}$ *the sequence of compact sets* $(s_{q^{-t}}(X_{[aq^t]}(\mathbf{r};(\frac{k_1}{l}, \ldots, \frac{k_s}{l});\omega)))_{t\geq 0}$ *converges with respect to the Hausdorff metric and*

$$A_q(r_1^{k_1}\cdots r_s^{k_s}) = \lim_{t\to\infty} s_{q^{-t}}(X_{[aq^t]}(\mathbf{r};(\frac{k_1}{l}, \ldots, \frac{k_s}{l});\omega)).$$

Proof

For $\omega = (\omega(n))_n \in \Omega^{\mathbb{N}}$ let

$$\omega^i(n) = \mathrm{card}\{k \mid \omega(n) = i, k \leq n\}.$$

Here we shall consider the polynomials

$$R_{n,\omega}(x) = \prod_1^s (r_i(x))^{\omega^i(n)} = \sum_\alpha \hat{s}(\alpha, n)x^\alpha$$

and the sequence $\mathcal{S}(\mathbf{r};\omega) = (R_{n,\omega})_{n\geq 0}$.

On the probability space (Ω, μ_θ) we consider the random variables $Y_n^j : \Omega \longrightarrow \{0,1\}$, defined by

$$Y_n^j(\omega) = \begin{cases} 1 & \omega(n) = j \\ 0 & \text{otherwise.} \end{cases}$$

The expected value of Y_n^j with respect to the measure μ_θ is $E(Y_n^j) = \theta_j$ and the variance $var(Y_n^j) = \theta_j(1-\theta_j) = \sigma_j^2$. Then $(Y_n^j)_{n\geq 0}$ is i.i.d. sequence of random variables.

The law of iterated logarithm, [12], p. 42, implies that for almost all points $\omega \in \Omega$ with respect to the Bernoulli measure μ_θ,

$$\limsup_n \frac{\sum_{k=1}^n Y_k^j - n\theta_j}{\sqrt{2\sigma_j^2 n \log\log n}} = 1.$$

Therefore for almost all $\omega \in \Omega$ there exist a natural number $N = N(\omega)$ such that

$$[n\theta_j] - [2\sqrt{2\sigma n \log\log n}] \leq \omega^j(n) \leq [n\theta_j] + [2\sqrt{2\sigma n \log\log n}]$$

for all $n \geq N$ and $\sigma = \max\{\sigma_j^2 \mid j = 1, \ldots, s\}$. Here we used that $\omega_j(n) = \sum_{k=1}^n Y_k^j(\omega)$.

Define the functions $g_{j,\omega} : \mathbb{N} \longrightarrow \mathbb{N}$ and $h : \mathbb{N} \longrightarrow \mathbb{N}$ with

$$g_{j,\omega}(n) = \omega_j(n),\ j = 1, \ldots, n,$$

and

$$h(n) = [2\sqrt{2\sigma n \log\log n}].$$

Then the functions h and $g_{j,\omega}, j = 1, \ldots, s$ satisfy the conditions of the Proposition 4. Therefore for almost all $\omega \in \Omega$ the sequence $\mathcal{S}(\mathbf{r}; \omega)$ is a small perturbation of the sequence $\mathcal{S}(\mathbf{r}; \theta)$ and the theorem follows from the Proposition 4.

□

8 Hausdorff dimension of rescaled evolution sets

The Hausdorff dimension of the rescaled evolution set $A_p(r) = A_p(r; 1; 1)$ was calculated by S. Willson in [22], using the Perron-Frobenius eigenvalue λ of the transition matrix (associated to the polynomial r), introduced by him: $dim_H A_p(r) = \frac{\ln \lambda}{\ln p}$. Later on F. v. Hassler at all., [7] described the rescaled evolution set $A_q(r)$ using the attractor of a special graph directed construction - matrix substitution system. The transition matrix of S. Willson coincides with the transition matrix of this graph directed construction. Since the graph directed construction of v. Haeseler et all. satisfies the open set condition, then the formula of S. Willson follows from the formula of Mauldin-Williams, [15], Theorem 3, for the Hausdorff dimension of the attractors of graph directed construction, [15]. In addition, from Mauldin-Williams, [15], Theorem 3, follows that the Hausdorff measure in the dimension is positive and finite, since the transition graph of the graph directed construction of Haeseler et all. is strongly connected, [9]. About the Hausdorff dimension see [3].

The goal of this section is the following proposition

Proposition 5 *Let $\mathbf{r} = (r_1, \ldots, r_s), r_1, \ldots, r_s \in \mathbb{F}_q[x], \theta_i, a \in \mathbb{R}, a > 0, \theta_i > 0, i = 1, \ldots, s$. Then*

a) The Hausdorff dimension of the rescaled evolution set $A_q(\mathbf{r}; \theta; a)$ does not depend on a, i.e.,

$$dim_H A_q(\mathbf{r}; \theta; a) = dim_H A_q(\mathbf{r}; \theta; 1).$$

b) For $s = 1, r_1 = r, \theta_1 = \theta$ follows

$$dim_H A_q(r; \theta; a) = dim_H A_q(r),$$

c) For $\theta_i = (\frac{k_1}{l}, \ldots, \frac{k_s}{l}), l, k_1, \ldots, k_s \in \mathbb{N}$ follows

$$dim_H A_q(\mathbf{r}; (\frac{k_1}{l}, \ldots, \frac{k_s}{l}); a) = dim_H A_q(r_1^{k_1} \cdots r_s^{k_s}).$$

d) For $\theta_i > 0, i = 1, \ldots, s$ and $\sum_1^s \theta_i = 1$, let μ_θ be the Bernoulli measure on the space $\Omega = \{1, \ldots, s\}^{\mathbb{N}}$. Then for almost all $\omega \in \Omega$ with respect to the Bernoulli measure μ_θ follows

$$dim_H A_q(\mathbf{r}; \theta; \omega; a) = dim_H A_q(\mathbf{r}; \theta; 1).$$

Proof

a) Recall that

$$A_q(\mathbf{r};\theta;a) = \lim_{t\to\infty} s_{q^{-t}}(X_{[aq^t]}(\mathbf{r};\theta)).$$

Set $\kappa = [\frac{\ln a}{\ln q}] + 1$. Then for t big enough $t_1 = t + \kappa$ is a nonnegative integer and

$$q^{t+\kappa-2} \leq aq^t \leq q^{t+\kappa}.$$

Therefore

$$X_{[q^{t+\kappa-2}]}(\mathbf{r};\theta) \subset X_{[aq^t]}(\mathbf{r};\theta) \subset X_{[q^{t+\kappa}]}(\mathbf{r};\theta).$$

The last inclusions imply

$$s_{q^{\kappa-2}}(A_q(\mathbf{r};\theta;1)) \subset A_q(\mathbf{r};\theta;a) \subset s_{q^\kappa}A_q(\mathbf{r};\theta;1)).$$

The assertion follows, since the affine maps do not change the Hausdorff dimension.

b) The assertion follows from the Proposition 3 and a).

c) The assertion follows from the Proposition 2 and a).

d) Follows from the Theorem 1 and a).

□

Remark 3 In general, the set $A_q(\mathbf{r};\theta;1)$ and its Hausdorff dimension depend on θ, e.g. in the case $\theta = (\frac{k_1 u}{l}, \ldots, \frac{k_i u}{l}), u, l, k_1, \ldots, k_s \in \mathbb{N}$ the Hausdorff dimension of the set $A_q(\mathbf{r};(\frac{k_1 u}{l}, \ldots, \frac{k_s u}{l}))$ depends in general on u.

The formula of S. Willson gives the dimension of the rescaled evolution set for θ vector with rational coordinates. For all other cases we do not have a formula for calculating it.

9 Some remarks

1. *The field of coefficients*

We considered polynomials with coefficients in the Galois field $\mathbb{F}_q$. All assertions proved in this note hold in a more general situation - for m-Fermat polynomials with coefficients in a commutative ring with 1.

Let R be a finite commutative ring with 1 and r a polynomial with coefficients in R. We say that the polynomial r has the m-Fermat property (or is an m-Fermat polynomial) if

$$r(x)^m = r(x^m)$$

for a given natural number $m \geq 2$.

Examples 3 1. For $R = \mathbb{F}_q$, the Galois field with $q = p^s$ elements (where p is a prime number), every polynomial $p(x) \in \mathbb{F}_q[x]$ is a q-Fermat polynomial, [10], pp. 62, 65.

2. For $R = \mathbb{Z}/p^s\mathbb{Z}$, ($p$ - prime number), the integers modulo p^s, $s \geq 2$, every polynomial $p(x) = q(x)^{p^{s-1}} \in R[x]$ is a p-Fermat polynomial, [17], [22].

3. For more examples and comments see [1], Lemma 1, 2 , pp. 11-12.

2. *Scaling sequences*

Here we shall consider the simplest case - the sequence of polynomials $\mathcal{S}(r)$ for $r \in \mathbb{F}_q[x]$.

The set of scaling sequences for a given polynomial is quite big: for arbitrary sequences $\underline{a} = (a(n))_{n\geq 0}$ and $\underline{b} = (b(n))_{n\geq 0}$ satisfying the conditions $dn\frac{b(n)}{a(n)} \leq C$, where $d = \deg r$ and C is an appropriate constant, the sets $s_{a(n)^{-1}}(X_{b(n)}(r))$ are subset of $[0,C]\times[0,C] = [0,C]^2$. The space $(\mathcal{K}([0,C]^2), \rho_H)$ is compact. Therefore there exist subsequences $\underline{a'} = (a(n_k))_{k\geq 0}$ and $\underline{b'} = (b(n_k))_{k\geq 0}$ such that the sequence $(s_{a(n_k)^{-1}})(X_{b(n_k)}(r)))_{n\geq 0}$ converges, i.e., the sequences $\underline{a'}$ and $\underline{b'}$ are scaling sequences for $X(\mathcal{S}(r))$ (or for the polynomial r). In general these scaling sequences depend on the polynomial r. We want to have scaling sequences independent on r. Such are the standard sequences $\underline{a} = (q^t)_{t\geq 0}$ and $\underline{a} = ([aq^t])_{t\geq 0}$ for every positive real number a. For $a = 1$ these are not only "universal" scaling sequences but the corresponding rescaled evolution set $A_q(r)$ have a self-similar structure generated by some special graf-directed system, [7]. Moreover, the rescaled evolution set $A_q(r;1;a)$ is determined by

$$A_q(r;1;a) = A_q(r) \cap (\mathbb{R} \times [0,a]), \text{ for } a < 1$$

and

$$A_q(r;1;a) = s_{q^{t_1+1}}(A_q(r)) \cap (\mathbb{R} \times [0,a]), \text{ for } q^{t_1} \leq a < q^{t_1+1}.$$

We shall call $A_q(r)$ the standard rescaled evolution set.

For some specific polynomials the self-similarity structure of the rescaled evolution set with respect to some other scaling sequences is simpler in comparison with the self-similarity structure of the standard rescaled evolution set. In [6] are given such examples. One of them is the following. Let $r(x) = 1 + x + x^2 \in \mathbb{F}_3[x]$. The sequences $\underline{a} = \underline{b} = (\frac{3^n-1}{2})_{n\geq 0}$ are scaling sequences for the polynomial r. Denote by B the rescaled evolution set with respect to these sequences:

$$B = \lim_{n\to\infty} s_{\frac{2}{3^n-1}}(X_{\frac{3^n-1}{2}}(r)).$$

The set B is not affinely equivalent with the standard rescaled evolution set $A_3(s), s \in \mathbb{F}_3[x]$ (the last set is the rescaled version of the geometrical representation (zero - non-zero) of the binomial coefficients modulo 3, [4]. The sets B and $A_3(r)$ are not homeomorphic and the self-similarity structure of B is simpler.

Similar observations imply that the sequences $\underline{a} = \underline{b} = (n)_{n\geq 0}$ are not scaling sequences for all polynomials with coefficients in $\mathbb{F}_2$. Let $r = 1 + x \in \mathbb{F}_2[x]$. The standard rescaling evolution set $A_2(r)$ of this polynomial is the Sierpinski triangle. Assume that the sequences $\underline{a} = \underline{b} = (n)_{n\geq 0}$ are scaling sequences for the polynomial r. Then

$$A_2(r) = \lim_{t\to\infty} s_{2^{-t}}(X_{2^t}(r)) = \lim_{t\to\infty} s_{3.2^{-t}}(X_{3.2^t}(r)) = s_3(\lim_{t\to\infty} s_{2^{-t}}(X_{3.2^t}(r))) = s_3(A_2(r;1;3)),$$

i.e., the sets $A_2(r)$ and $A_2(r;1;3)$ are affinely equivalent, which is not possible since they are not homeomorphic.

At the end we shall prove that the sequences $\underline{a} = \underline{b} = (3^n)_{n\geq 0}$ are not scaling sequences for the polynomial $r(x) = 1 + x \in \mathbb{F}_2[x]$.

The evolution set $X(r)$ of this polynomial is the geometrical representation (zero - non-zero) of the sequence of binomial coefficients modulo 2 and its rescaled evolution set

$$A_2(r) = \lim_{n\to\infty} s_{2^{-n}}(X_{2^n}(r))$$

is the Sierpinski triangle.

The number $\xi = \frac{\log 2}{\log 3}$ is irrational and its simple continued fraction $\xi = [0; a_1, a_2, \ldots]$ is infinite. Let

$$\frac{p_k}{q_k} = [0; a_1, \ldots, a_k],\ \gcd(p_k, q_k) = 1$$

be the k-th convergent of the continued fraction $[0; a_1, a_2, \ldots]$, [10], Ch. X.

Then

$$|\xi q_k - p_k| \leq \frac{1}{q_k},\ \text{for}\ k = 1, 2, \ldots,$$

[10], p. 140. The last inequality implies

$$|1 - \frac{2^{q_k}}{3^{p_k}}| \leq \frac{2}{q_k}, \tag{54}$$

$$\frac{3^{p^k}}{2^{q^k}} \leq 1 + \frac{2}{q_k}, \tag{55}$$

for $k = 1, 2, \ldots$. In [6] is proved

$$A_2(r) = \lim_{k\to\infty} s_{2^{-q_k}}(X_{2^{q_k}}(r)) = \lim_{k\to\infty} s_{3^{-p_k}}(X_{3^{p_k}}(r)). \tag{56}$$

Since $\lim q_k = \infty$ this assertion follows from the following inequalities

$$\rho_H(s_{2^{-q_k}}(X_{2^{q_k}}(r)), s_{3^{-p_k}}(X_{3^{p_k}}(r))) \leq$$

$$\tfrac{1}{2^{q_k}}\rho_H(X_{2^{q_k}}(r), X_{3^{p_k}}(r)) + \tfrac{1}{2^{q_k}}\rho_H(X_{3^{p_k}}(r), s_{2^{q_k}3^{-p_k}}(X_{3^{p_k}}(r))) \leq$$

$$\tfrac{d}{2^{q_k}}|2^{q_k} - 3^{p_k}| + \tfrac{1}{2^{q_k}}|1 - \tfrac{2^{q_k}}{3^{p_k}}|diam X_{3^{p_k}}(r) \leq (1 + \tfrac{2}{q_k})\tfrac{4d}{q_k}.$$

Assume that the sequence $(s_{3^{-n}}(X_{3^n}(r)))_{n\geq 0}$ converges. Then from (56) follows that its limit is the Sierpinski triangle $A_2(r)$.

We know that the sequence $(s_{2^{-n}}(X_{3.2^n}(r)))_{n\geq 0}$ converges and its limit is the set

$$A_2(r; 1; 3) = s_4(A_2(r)) \cap ([0, 3] \times [0, 3]).$$

The last set is not homeomorphic to the Sierpinski triangle.

From (54) follows

$$\rho_H(X_{3.2^{q_k}}(r), (X_{3^{p_k+1}}(r)) \leq 3d|2^{q_k} - 3^{p_k}| \leq \frac{2d3^{p_k+1}}{q_k}.$$

Therefore, the sequence $(s_{3^{-p_k}}(X_{3.2^{q_k}}(r)))_{k\geq 0}$ converges and

$$\lim_{k\to\infty} s_{3^{-p_k}}(X_{3.2^{q_k}}(r)) = \lim_{k\to\infty} s_{3^{-p_k}}(X_{3^{p_k+1}}(r)) = s_3(A_2(r))$$

(since the sequence $(s_{3^{-p_k-1}}(X_{3^{p_k+1}}(r)))_{k\geq 0}$ is a subsequence of $(s_{3^{-n}}(X_{3^n}(r)))_n$).

From (54) and (55) follows

$$\rho_H(s_{2^{-q_k}}(X_{3.2^{q_k}}(r)), s_{3^{-p_k}}(X_{3.2^{q_k}}(r))) = \frac{1}{2^{q_k}}\rho_H(X_{3.2^{q_k}}(r), s_{2^{q_k}3^{-p_k}}(X_{3.2^{q_k}}(r))) \leq$$

$$\frac{1}{2^{q_k}}|1-\frac{2^{q_k}}{3^{p_k}}|diam X_{3^{p_k}}(r) \leq \frac{6d}{q_k},$$

therefore

$$A_2(r;1;3) = \lim_{k\to\infty} s_{2^{-q_k}}(X_{3.2^{q_k}}(r)) = \lim_{k\to\infty} s_{3^{-p_k}}(X_{3^{p_k+1}}(r)) = s_3(A_2(r)),$$

i.e., the sets $A_2(r;1;3)$ and $A_2(r)$ are affinely equivalent, which is impossible since they are not homeomorphic.

With similar arguments it follows that the sequences $\underline{a} = \underline{b} = (m^n)_{n\geq 0}, m \geq 2$ are not scaling sequences for the polynomial $r(x) = 1 + x \in \mathbb{F}_2[x]$ in the case the natural number m is not a power of 2. This is a geometrical counterpart of the result of Allouche et all, [1], that the sequence of binomial coefficients $(\binom{n}{k} \bmod 2)_{n,k}$ is m-automatic if and only if m is a power of 2.

3. *Polynomials in k-variables*

All results of this note also hold for polynomials $r \in \mathbb{F}_q[x_1,\ldots,x_k]$. Then the rescaled evolution sets are fractal subsets in the $k+1$-dimensional Euclidean space $\mathbb{R}^{k+1}$. For the standard scaling seqiences $\underline{a} = \underline{a} = (q^t)_{t\geq 0}$ and the parameters $a = \theta_1 = \cdots = \theta_s = 1$ this is proved in [6]. For example the standard rescaled evolution set $A_2(r)$ of the polynomial $r(x_1,x_2) = 1 + x_1 + x_2 \in \mathbb{F}_2[x_1,x_2]$ is the Sierpinski pyramid, [14].

4. *Hausdorff dimension of rescaled evolution sets does not depend on scaling sequences*

The squeezing trick of S. Willson, [23], applied in section 8 implies: if $\underline{a} = \underline{b} = (a(n))_{n\geq 0}$ are scaling sequences and

$$A_{\underline{a}}(\mathbf{r};\theta) = \lim_{n\to\infty} s_{a(n)^{-1}}(X_{a(n)}(\mathbf{r};\theta)),$$

then

$$dim_H A_{\underline{a}}(\mathbf{r};\theta) = dim_H A_q(\mathbf{r};\theta;1).$$

S. Willson proved this for $s = 1, \theta_1 = 1$ with the following argument. For every n choose the natural number k_n such that $q^{k_n} \leq a(n) \leq q^{k_n+1}$, then

$$X_{q^{k_n}}(\mathbf{r};\theta) \subset X_{a(n)}(\mathbf{r};\theta) \subset X_{q^{k_n+1}}(\mathbf{r};\theta),$$

and

$$s_{q^{-k_n-1}}(X_{q^{k_n}}(\mathbf{r};\theta)) \subset s_{a(n)q^{-k_n-1}}(s_{a(n)^{-1}}(X_{a(n)}(\mathbf{r};\theta))) \subset s_{q^{-k_n-1}}(X_{q^{k_n+1}}(\mathbf{r};\theta)). \qquad (57)$$

By choosing a suitable subsequence if necessary we may assume that $\lim_{n\to\infty} \frac{a(n)}{q^{-k_n-1}} = \lambda$. By taking limit in (57) we obtain

$$s_{q^{-1}}(A_q(\mathbf{r};\theta)) \subset s_\lambda(A_{\underline{a}}(\mathbf{r};\theta)) \subset A_q(\mathbf{r};\theta),$$

which implies the assertion.

10 Open questions

Let $r_1, \dots, r_s$ be polynomials with the coefficients in the Galois field $\mathbb{F}_q$ and let $\theta_1, \dots, \theta_s$ be positive real numbers.

- Is the Hausdorff dimension $\phi(\theta_1, \dots, \theta_s) = dim_H A_q(\mathbf{r}; \theta; 1)$ a continuous function on $\theta = (\theta_1, \dots, \theta_s)$ for a fixed $\mathbf{r} = (r_1, \dots, r_s)$?
- Is there a simple formula for the Hausdorff dimension of $dim_H A_q(\mathbf{r}; \theta; 1)$?

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