DIMENSION AND MEASURES FOR A CURVILINEAR SIERPINSKI GASKET OR APOLLONIAN PACKING

by

R. Daniel Mauldin[1]
and
Mariusz Urbański[1]

Department of Mathematics
University of North Texas
Denton, Texas 76203

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ABSTRACT. In this paper we apply some results about general conformal iterated function systems to A, the residual set of a standard Apollonian packing or a curvilinear Sierpinski gasket. Within this context, it is straight forward to show that h, the Hausdorff dimension of A is greater than 1 and the packing dimension and the upper and lower box counting dimensions are all the same as the Hausdorff dimension. Among other things, we verify Sullivan's result that $0 < \mathcal{H}^h(A) < \infty$ and $\mathcal{P}^h(A) = \infty$.

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§1. Introduction: Setting and Notation

The purpose of this note is to demonstrate how the theory of infinite systems of conformal maps can be applied to obtain some results about the dimension and measure of the A, the residual set of a standard Apollonian packing or, equivalently a curvilinear Sierpinski gasket. First, let us describe the setting.

Let $X = B(0,1)$ and let $f = \frac{(\sqrt{3}-1)z+1}{-z+\sqrt{3}+1}$. Then $f(z_j) = a_j$, where $z_j = e^{\frac{2\pi ij}{3}}$ for $j = 0,1,2$ and $a_0 = 1, a_1 = (2-\sqrt{3})(1/2+\sqrt{3}i/2)$ and $a_2 = (2-\sqrt{3})(1/2-\sqrt{3}i/2)$. Let $R_1(z) = e^{\frac{2\pi i}{3}}z$ and $R_2(z) = R_1^2(z)$. Let $f_1 = f, f_2 = R_1 \circ f$, and $f_3 = R_2 \circ f$. Let

$$A = \bigcap_n \bigcup_{|\sigma|=n} f_\sigma(X),$$

where $\sigma = (s_1, s_2, ..., s_n) \in \{0,1,2\}^n$ and $f_\sigma = f_{s_1} \circ \cdots \circ f_{s_n}$. The set X and some of its images are indicated in figure 1.

FIGURE 1 GOES HERE

Then A is the limit set generated by the finite iterated function system $\{f_i\}_{i=1}^3$ and A satisfies the self-conformal set equation:

$$A = \bigcap_n \bigcup_{|\sigma|=n} f_\sigma(A).$$

Now A is also the residual set generated from the Apollonian packing or the osculatory packing of the curvilinear equilateral triangle, T with vertices z_0, z_1, z_2. This is clear since $f_i(T) \subset T$, and $\cup_i f_i(T)$ consists of T with the inscribed circle removed, in general, $\cup_i f_{\sigma i}(T)$, is the curvilinear triangle $f_\sigma(T)$ with the inscribed circle removed and A can be expressed as

$$A = \bigcap_n \bigcup_{|\sigma|=n} f_\sigma(T).$$

The set T and some of its images are indicated in figure 2.

FIGURES 2a AND 2b GO HERE. ONE ABOVE THE OTHER

One of the problems in analyzing the geometric properties of A has been the fact that although the finite system of conformal maps f_i satisfies the open set condition, the maps are not contractive but only nonexpansive since there is a neutral fixed point and also the system does not satisfy the bounded distortion property. Thus, we cannot apply the theory that has been developed for self conformal sets generated by finitely many uniformly contracting conformal maps satisfying bounded distortion. In fact, the residual set A cannot be generated be any finite family of uniformly contractive conformal maps which satisfy the open set condition and bounded distortion. The reason is that if this were the case, then both the Hausdorff and packing measure of A in its dimension would be positive and finite [MU1]. This would conflict with Sullivan's result that although the Hausdorff measure of A is positive and finite, the packing measure is infinite [S]. Our goal in this paper is to show how this result and some others can be obtained by modifying the system. Specifically, we will show that by deleting a countable set from A, we obtain a set which is the limit set generated by an infinite family of uniformly contracting conformal maps and this family satisfies the required conditions for analysis of an infinite iterated function system. It is within this context that we show $0 < \mathcal{H}^h(A) < \infty$ and $\mathcal{P}^h(A) = \infty$ where $h = dim_H(A) = dim_P(A)$. We note that the packing measure we use is not the "packing" measure as defined in Sullivan's paper but the now standard packing measure defined by Taylor and Tricot [TT],[M]. We show h is also the upper and lower box counting dimension of A. It is shown in [MU3] that the conformal measure for the modified infinite system is also conformal for the original system. However, the equivalent invariant measure for the modified system is not invariant for the original system, but as indicated here can be adjusted to give an invariant measure for the original one. Finally, we note that McMullen has given an algorithm for computing h [Mc].

Let us describe the family of maps forming the infinite conformal iterated function system. Let $I = \{(n,j) : j, n \in \mathbb{N} \text{ and } 1 \le n \le 6\}$. Let $\phi_{1,j} = f^j \circ R_1 \circ f$, $\phi_{2,j} = f^j \circ R_2 \circ f$, $\phi_{3,j} = R_1 \circ f^j \circ R_1 \circ f$, $\phi_{4,j} = R_1 \circ f^j \circ R_2 \circ f$, $\phi_{5,j} = R_2 \circ f^j \circ R_1 \circ f$, and $\phi_{6,j} = R_2 \circ f^j \circ R_2 \circ f$.

This system satisfies all the requirements to be an infinite conformal iterated function system as described in [MU1, MU2]. The bounded distortion property is satisfied by the Koebe distortion lemma. Figure 3 indicates some of the images of X under this family. The limit set generated by this family of maps is $J = A \setminus C$, where C is the countable set of cusp points of A.

FIGURE 3 GOES HERE

§2. Results

First, we need to estimate the size of the derivatives of the maps in our family. Let $g(z) = 1/z - 1$. Then $g^{-1}(z) = 1 + 1/z$ and $h(z) = g \circ f \circ g^{-1}(z) = z - 1/\sqrt{3}$. Thus, $h^n(z) = z - n/\sqrt{3}$ and $f^n(z) = \frac{(\sqrt{3}-n)z+n}{-nz+n+\sqrt{3}}$. From this we have $(f^n)'(z) = \frac{3}{(-nz+n+\sqrt{3})^2}$.

From these formulas the following lemma can be proven.

Lemma 2.1. *There is a constant $Q > 1$ such that for all $(n,j) \in I$,*

$$Q^{-1}/j^2 \le ||\phi'_{n,j}|| \le Q/j^2.$$

Let

$$\psi_n(t) = \sum_{\omega \in I^n} ||\phi'_\omega||^t.$$

and

$$P(t) = \lim_{n\to\infty} \frac{1}{n} \log \psi_n(t).$$

Thus, P is the topological pressure function for this system. From lemma 2.1, we have that $\psi_1(1/2) = \infty$ and if $t > 1/2$, then $\psi_1(t) < \infty$. Therefore, this system is strongly regular as described in [MU2]. This implies there is some $h > 1/2$ such that $P(h) = 0$. We will prove that h is the Hausdorff and packing dimension of J and that there is an h-conformal probability measure, m, supported on A for this system [MU1].

Our first result is a simple proof of the following well-known result, see [F], pp. 125-131.

Theorem 2.2. $1 < dim_H(A) < 2$.

Proof. Let us note that $\mathcal{H}^1(A) > 0$, since A is a continuum. We give a topological argument that the Hausdorff dimension must be greater than 1. Let us assume to the contrary that the dimension is 1. From the results of [MU1] we know that $\mathcal{H}^1(A) < \infty$. However, in order for a continuum to have finite $\mathcal{H}^1$ measure with respect to some compatible metric, the continuum must have uncountably many local separating points. In fact, in order for this to be so every nondegenerate subcontinuum must contain uncountably many local separating points [EH]. However, A has only countably many local separating points-those points which are cusp points at some level. This contradiction allows us to conclude that $1 < dim_H(A)$.

To see that $dim_H(A) < 2$, note that $\lambda_2(Int(X)\backslash \cup_{i,n\in I}\phi_{i,n}(X)) > 0$. So, by theorem 4.5 of [MU1], $h < 2$. The proof is finished. ■

Remark. The local separating point argument also allows us to conclude that A does not have σ-finite $\mathcal{H}^1$ measure [M]. This topological argument does not give us any means of estimating how much greater than 1 the dimension of A is whereas the arguments of Hirst and Boyd as presented in Falconer's book [F] do.

Our next aim is to show that the Hausdorff, upper and lower box counting, and packing dimensions of A are equal. We begin with the following lemma.

Lemma 2.3. *If z is in the open segment joining 0 and 1, then $\overline{dim}_{\mathrm{B}}(O(z)) \le 1$, where $O(z) = \{\phi_{i,n}(z) : i \le 6, n \ge 1\}$.*

Proof. Of course, it suffices to show that $\overline{dim}_{\mathrm{B}}(\{f^n(z) : n \ge 1\} \le 1$. Put

$$r_n = \frac{\sqrt{3}(1-z)}{(1-z)n + \sqrt{3}}.$$

Since the sequence r_n decreases to 0, given r sufficiently small there exists exacly one $n = n(r)$ such that

1 $$r_{n+1} \le r \le r_n$$

Let $N_s(Z)$ denote the minimal number of balls of radius s needed to cover the set Z. Notice that $r_k = 1 - f^k(z)$. and therefore all the points $f^k(z)$, $k \ge n+1$, are covered by the ball $B(f^{n+1}(z), r_{n+1})$. Hence $N_r(O(z)) \le N_{r_{n+1}}(O(z)) \le n+2$. Thus, by (1)

$$\frac{\log N_r(O(z))}{-\log r} \le \frac{\log(n+2)}{-\log r_n} = \frac{\log(n+2)}{\log(n(1-z)+\sqrt{3}) - \log(\sqrt{3}(1-z))} \to 1$$

if $n \to \infty$ or equivalently, if $r \to 0$. Thus $\overline{dim}_{\mathrm{B}}(O(z)) \le 1$ and the proof is completed. ■

Invoking now Theorem 2.0 and Theorem 2.11 from [MU2], we get the following.

Theorem 2.4. *The Hausdorff, upper and lower box counting, and packing dimensions of A are equal.*

Let us fix some notation. Also, let $a_j = \phi_{1,j}(z_1), b_j = \phi_{1,j}(z_0)$, and $c_j = \phi_{1,j}(z_2)$. So a_j, b_j, c_j are the vertices of the triangle $\phi_{1,j}(T)$ arranged such that $a_j, b_j \in T$ and $c_j \notin T$.

Lemma 2.5 *There is a constant $C > 1$ such that if k and n are positive integers with $k+1 < n, z \in \phi_{1,k}(X)$, and $y \in \phi_{1,n}(X)$, then $C(\frac{1}{k} - \frac{1}{n}) \ge |z-y| \ge C^{-1}(\frac{1}{k} - \frac{1}{n})$.*

Proof. Let y'(z') be the point of intersection of the real axis and the line, $L_1, (L_2)$ through y(z) and $1+\sqrt{3}i$. Let θ be the angle between the lines L_1 and L_2. Let $\alpha(\alpha')$ be the angle between L_2 and the line through y and z(the real axis). By the law of sines, we have

$$\frac{|z-y|}{\sin\theta} = \frac{|1+i\sqrt{3}-y|}{\sin\alpha}; \frac{|z'-y'|}{\sin\theta} = \frac{|1+i\sqrt{3}-y'|}{\sin\alpha'}.$$

Thus,

$$|z-y| = |z'-y'|\frac{|1+i\sqrt{3}-y|}{|1+i\sqrt{3}-y'|}\frac{\sin\alpha'}{\sin\alpha}.$$

Clearly, $|1+i\sqrt{3}-y| \le |1+i\sqrt{3}-y'| \le 2|1+i\sqrt{3}-y|$ and α and α' are bounded away from 0. For each $n \ge 0$, let u_n be the center of the circle $f^n(B(0, 2-\sqrt{3}))$ Also, note that for each $p \ge 1$, the line through $1+i\sqrt{3}$ and $a_p(b_p)$ which is tangent to the disk $\phi_{1,p}(X)$ meets the real axis at u_p. Thus, there is some $M > 1$ such that $M^{-1}(u_{n-1} - u_k) \le |z'-y'| \le M(u_n - u_{k-1})$.
Let $v_0 = \sqrt{3}-2$ and $v_1 = f(v_0) = 2-\sqrt{3}$. Then for each $n \ge 0, u_n = (f^n(v_0) + f^{n+1}(v_0))/2$. From this it follows that there is a constant $D > 1$ such that if k and n are positive integers with $k+1 < n$, then $D^{-1}(\frac{1}{k} - \frac{1}{n}) \le u_n - u_k \le D(\frac{1}{k} - \frac{1}{n})$. The lemma now follows. ■

Let $H > 0$ be such that if k and n are positive integers with $k < n$, then

$$\sum_k^n \frac{1}{j^{2h}} \le H[(\frac{1}{k})^{2h-1} - (\frac{1}{n})^{2h-1}].$$

Theorem 2.6. $0 < \mathcal{H}^h(J) < \infty$.

Proof. That $\mathcal{H}^h(J) < \infty$ follows immediately from Lemma 4.2 of [MU1]. Let $F = \{(i,n) \in I : n = 1, 2\}$. We shall show if L and $\gamma \ge 1$ are large enough, then for all $(i,n) \in I \setminus F$, for all $r > \gamma \mathrm{diam} \phi_{i,n}(X)$, and for all $y \in \phi_{i,n}(X)$ we have:

$$m(B(y,r)) \le Lr^h. \tag{2.1}$$

It then follows from lemma 4.11 of [MU1] or theorem 2.4 of [MU2]) that $0 < \mathcal{H}^h(J)$. We note that we only need to prove that (2.1) holds for sufficiently small r. Since our system is symmetric with respect to rotations by the angles $2\pi/3$ and $4\pi/3$, and with respect to reflections about the real axis and the lines passing through the origin and the point $\mathrm{e}^{\frac{2\pi i}{3}}$ or the point $\mathrm{e}^{\frac{4\pi i}{3}}$, it suffices to consider the sets $\phi_{1,n}(X)$. Choose γ such that if $r > \gamma \mathrm{diam} \phi_{1,n}(X)$, then $r \ge 1/n^2$ and $r > max\{|y-z| : y \in \phi_{1,n}(X), z \in \phi_{1,m}(X), |m-n| = 1\}$. Now, fix $n \ge 3$, $y \in \phi_{1,n}(X)$, and a radius $r \ge \gamma \mathrm{diam}(\phi_{i,n}(X)) \ge 1/n^2$. Let $k \le n$ be the least positive integer such that $\phi_{1,k}(X) \cap B(y,r) \ne \emptyset$ and choose $z \in \phi_{1,k}(X) \cap B(y,r)$.

By lemma 2.5, we have

$$r \ge |z - y| \ge C^{-1}(\frac{1}{k} - \frac{1}{n}).$$

Hence $\frac{1}{k} \le Cr + \frac{1}{n}$. By $m\_B(y,r)$, we denote the measure of the union of all the sets $\phi_{1,j}(X)$ that intersect $B(y,r)$ and for which $j \le n$. With this notation, we have

$$\begin{aligned} m\_B(y,r) &\le \sum_k^n \frac{1}{j^{2h}} \le H[(\frac{1}{k})^{2h-1} - (\frac{1}{n})^{2h-1}] \\ &\le H[(Cr + \frac{1}{n})^{2h-1} - (\frac{1}{n})^{2h-1}]. \end{aligned}$$

We shall consider now two cases.

Case 1. $r \ge 1/n$. We make the estimate:

$$m_-(B(y,r)) \le H(Cr + r)^{2h-1} \le H(C+1)^{2h-1} r^{h-1} r^h.$$

Since $h - 1 > 0$, the right hand side is less than r^h for r small enough which is all we need.

Case 2. $r \le 1/n$. Then, recalling that $r \ge 1/n^2$ and using the Mean Value theorem, there exists θ, $\frac{1}{n} \le \theta \le \frac{1}{n} + Cr \le \frac{1}{n}(C+1)$ such that

$$m_-(B(y,r)) \le H(2h-1)\theta^{2h-2}Cr = H(2h-1)(\theta^2)^{h-1}Cr$$
$$\le H(2h-1)(\frac{1}{n^2})^{h-1}(C+1)^{2h-2}Cr \le CH(2h-1)(C+1)^{2h-2}r^h.$$

Now, consider $m_+B(y,r)$, the measure of the union of all the sets $\phi_{1,j}(X)$ that intersect $B(y,r)$ and for which $j \ge n$. First, suppose $r \ge 1/Cn$. We make the estimate:

$$m_+B(y,r) \le \sum_{j=n}^{\infty} \frac{1}{j^{2h}} \le H(\frac{1}{n})^{2h-1} \le HC^{2h-1}r^{h-1}r^h.$$

Again, for r sufficiently small, $m_+B(y,r) \le r^h$. Finally, suppose $r < 1/Cn$. Then $1 \notin B(y,r)$. Let k be the greatest integer such that $\phi_{1,k}(X) \cap B(y,r) \ne \emptyset$. By lemma 2.5, $r \ge |y-z| \ge C^{-1}(\frac{1}{n} - \frac{1}{k})$. Hence, $\frac{1}{k} \ge \frac{1}{n} - Cr$. We can now make the estimate:

$$m_+B(y,r) \le \sum_{j=n}^{k} \frac{1}{j^{2h}} \le H[(\frac{1}{n})^{2h-1} - (\frac{1}{k})^{2h-1}].$$

Using the fact that $r \ge 1/n^2$ and the Mean Value Theorem, there is some z with $\frac{1}{n} - Cr \le \frac{1}{k} \le z \le \frac{1}{n}$ such that

$$m_+B(y,r) \le H(2h-1)z^{2h-2}Cr \le CH(2h-1)(\frac{1}{n})^{2h-2}r \le CH(2h-1)r^{h-1}r = CH(2h-1)r^h.$$

The proof is finished. ■

Theorem 2.7. $\mathcal{P}^h(J) = \infty$.

Proof. We will show that

$$\lim_{r\to 0} \frac{m(B(1,r))}{r^h} = 0.$$

It then follows from lemma 4.12 of [MU1] that $\mathcal{P}^h(J) = \infty$.

Let $r < \sqrt{3}-1$ and let n be the smallest positive integer such that $\phi_{1,n}(X) \cap B(1,r) \ne \emptyset$. We have the inequalities:

$$\frac{m(B(1,r))}{r^h} \le \frac{1}{r^h}\{\sum_{k=n}^{\infty} m(\phi_{1,k}(X)) + \sum_{k=n}^{\infty} m(\phi_{2,k}(X))\}$$
$$\le \frac{2Q^h}{r^h} \sum_{k=n}^{\infty} \frac{1}{k^{2h}} \le \frac{2Q^h}{r^h} H \frac{1}{n^{2h-1}}.$$

By lemma 2.5 and the fact that $r > |z-1| > \frac{1}{Cn}$, we have

$$\frac{m(B(1,r))}{r^h} \leq \frac{2Q^h}{r^h} HC^{2h-1} r^{2h-1} \leq 2Q^h HC^{2h-1} r^{h-1}.$$

Since $h > 1$, the limit in question is zero and $\mathcal{P}^h(A) = \infty$. ■

Let Y denote the set of those x for which $\pi^{-1}(x)$ is a singleton, where π is the natural projection from the shift space $\{f_1, f_2, f_3\}^\infty$ onto the limit set. Write $\pi^{-1}(x) = f_1^n\omega$, where $n \geq 0$ and $\omega_1 \neq f_1$ and set $n(x) = n$. For every integer $n \geq 0$, set $B_n = \{x : n(x) = n\}$ and $D_n = \{x : n(x) \geq n\}$. Let μ denote the invariant probability measure for the modified system. Two proofs are given for the existence of the measure μ in [MU1], theorem 3.8 and lemma A.1. Then we have the following theorem which is proved in [MU3].

Theorem 2.8. *The conformal measure m for the modified infinite system, $\{\phi_{i,n}\}$ is also a conformal measure for the original system $\{f_1, f_2, f_3\}$. The measure given by the formula*

$$\nu(E) = \sum_{k=0}^{\infty} \sum_{|\omega|=k} \mu(f_\omega(E) \cap D_k)$$

defines a σ-finite measure equivalent with m and invariant under the original system generated by the maps f_1, f_2, f_3. Moreover, one can check that the measure ν is finite.

Let us comment some more on the system $\{f_1, f_2, f_3\}$. This system is an example of a general theory developed in [MU3]. First, let $P_0(t)$ be the pressure function for the original system$\{f_i\}_{i=1}^3$. Thus,

$$P_0(t) = \lim_{n\to\infty} \frac{1}{n} \log \psi_{0,n}(t),$$

where

$$\psi_{0,n}(t) = \sum_{\omega \in \{1,2,3\}^n} ||f'_\omega||^t.$$

Let $g : \Omega \to I\!R$ be defined by $g(\omega) = t\log|f'_{\omega_1}(\pi(\sigma(\omega)))|$. As shown in [MU3], a second expression for $P_0(t)$ is given by

$$P_0(t) = \lim_{n\to\infty} \sum_{|\tau|=n} \sup_{\{\omega\in[\tau]\}} e^{S_n g(\omega)}.$$

As is well known, see [W], a third expression for $P_0(T)$ is given by

$$P_0(t) = \sup\left\{ h_\nu(\sigma) + \int g(\omega)d\nu(\omega) : \nu \text{ is invariant under } \sigma \right\}.$$

From these two equivalent ways of expressing P_0, we have $P_0(0) = \log 3$, the function P_0 is continuous, nonincreasing, convex and Lipschitz continuous. Also, $dim_H(A) = \min\{t :$

$P_0(t) \le 0\}$. Thus, $P_0(t) > 0$, if $0 \le t < h$. If we take κ to be point mass at the infinite sequence of 1's, then κ is invariant and $h_\kappa(\sigma) + \int g(\omega) d\kappa(\omega) = 0$. Thus, $P_0(t) = 0$ if $t > h$, whereas $P(t) < 0$, if $t > h$.

We would like to indicate how some other features of the Apollonian packing can be obtained from this viewpoint. We begin with a theorem of D. Boyd [Bo], another proof of which was given by Tricot [T]. For some recent related work see [R].

Theorem 2.9 *Let $\{B_n\}_{n=1}^\infty$ be the disks that are removed from T to obtain the residual set A. Then $dim_H(A) = b = \inf\{e : \sum_{n=1}^\infty diam(B_n)^e < \infty\}$.*

Indication of the Proof. Notice the balls removed from T consist of $B = B(0, 2 - \sqrt{3})$ together with all the balls, $f_\omega(B)$. If $t<h$, then $\sum_{n=1}^\infty \sum_{\omega\in\{1,2,3\}^n} (diam f_\omega(B))^t \ge \sum_{\omega\in I^*} (diam \phi_\omega(B))^t \ge (2K(2-\sqrt{3}))^{-t} \sum_{\omega\in I^*} ||\phi'_\omega||^t \ge (2K(2-\sqrt{3}))^{-t} \sum_{n=1}^\infty \sum_{\omega\in I^n} ||\phi'_\omega||^t$, where I^* consists of all finite words in the alphabet I. But, for large n, $\sum_{\omega\in I^n} ||\phi'_\omega||^t > e^{nP(t)/2}$. This implies $\sum_{n=1}^\infty diam(B_n)^t = \infty$ and $h \le b$.

On the other hand, if $e > h$, then $P_0(e) = 0$ and there is an e-conformal measure m_e, a probability measure supported on A. This is proven in [MU3]. It turns out that m_e is supported on $\cup_{n=1}^\infty \cup_{|\omega|=n} f_\omega(1)$. Let G be an open ball such $B \subset G$ and $\overline{G}$ is a subset of the interior of X. So, $0 < m_e(G)$. Since G is bounded away from the unit circle, the family of maps f_ω have bounded distortion with some distortion constant $\overline{K}$. We have $m_e(f_\omega(G)) = \int_G |f'_\omega|^e dm_e \ge \overline{K}^{-e} ||f'_\omega||_G^e m_e(G)$, where $||\cdot||_G$ is the uniform norm over G. Thus, $\sum_{\omega\in\{1,2,3\}^*} (diam f_\omega(B))^e \le \sum_{\omega\in\{1,2,3\}^*} (diam f_\omega(G))^e \le diam(G)^e \sum_{\omega\in\{1,2,3\}^*} ||f'_\omega||_G^e \le diam(G)^e \overline{K}^e m_e^{-1}(G) \sum_{\omega\in\{1,2,3\}^*} m_e(f_\omega(G))$. But the sets $f_\omega(G)$ being disjoint, $\sum_{n=1}^\infty diam(B_n)^e < \infty$ and $b \le h$. ■

A general theory of families of conformal maps such as the family $\{f_i\}_{i=1}^3$ which generate the Apollonian packing are in [MU3].

Let us compare the conformal measure, m_G, for the standard Sierpinski gasket and the conformal measure for the curvilinear gasket. The standard gasket, G, is the limit set determined by three similarity maps S_1, S_2, S_3 with the same reduction ratio, 1/3. The Hausdorff dimension is $d = \log 3/\log 2$. We also have that $0 < \mathcal{H}^d(G) < \mathcal{P}^d(G) < \infty$. The conformal measure and equivalent invariant measure are equal. The conformal measure can also be realized of course as the uniform distribution on G. In other words, $m_G = \tau \circ \pi^{-1}$, where π is the natural projection map from the coding space $\Omega = \{1,2,3\}^N$ and τ is the uniform measure on the coding space or infinite product measure determined by the probability vector $(1/3, 1/3, 1/3)$. Thus, each set of the form $S_\sigma(G)$ has measure $1/3^n$ where $n = |\sigma|$. Now, the corresponding image γ of τ on the curvilinear gasket has been used to obtain a lower bound on the dimension of the residual set, see [F]. What one has of course is that $dim_H(\gamma) \le dim_H(A)$ and $dim_H(\gamma) = h_\tau/\chi_\tau$, where h_τ is the entropy of τ and $\chi_\tau = -\int_\Omega \log|f'_{\omega_1}(\pi(\sigma(\omega))|d\tau(\omega)$. Since the coding map is finite-to-one, $h_\gamma = h_\tau = \log 3$. It is natural to ask whether, as is the case with the standard gasket, γ is the invariant measure, or γ equivalent to the conformal measure m or even if $dim_H(\gamma) = dim_H(A)$. We give a partial answer in the next theorem.

Theorem 2.10 *Let $p = (p_1, p_2, p_3)$ be a probability vector, let τ be the corresponding infinite product measure on $\Omega = \{1, 2, 3\}^N$ and let γ be image measure on A induced by the coding map $\pi : \Omega \to A$. Then $\nu \neq \gamma$.*

Proof. Let ω be the infinite sequence of 1's. Then $\gamma(f_{\omega|n}(T)) = p_1^n$. Also, $\nu(f_{\omega|n}(T))$ has a bigger order than $m(f_{\omega|n}(T))$. As we have shown earlier, this last quantity is of the same order as n^{1-2h}. Thus, γ puts too little mass near the cusp points in comparison with ν. ■

In fact we believe the next conjecture should have a proof following the approach given by Ledrappier [Le1],[Le2].

C0njecture 2.11 *Let $\tilde{\gamma}$ be an invariant ergodic measure on Ω and let γ be the image measure on A. Then $dim_H(\gamma) < dim_H(A)$ unless $\gamma = \nu$, the unique invariant measure equivalent to the conformal measure m.*

Remark. In [F], p. 130, it is mentioned that $dim_H(A) \geq \log 3/\log\lambda$, where $\log\lambda$ is the Lyapunov exponent of the uniform distribution on Ω. The truth of the conjecture would allow us to conclude that $dim_H(A) > \log 3/\log\lambda$.

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