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On the Baire system generated by a linear lattice of functions

by

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Suppose G is a linear space of real functions defined over a point set S such that if f is in G, then $|f|$ is in G; G is a linear lattice of functions over S. Also, suppose G contains the constant functions over S. Let $B_0(G) = G$ and for each ordinal number α, $0<\alpha<\Omega$, let $B_\alpha(G)$ denote the collection of all pointwise limits of sequences from the collection $\sum_{\gamma<\alpha} B_\gamma(G)$. Sierpiński [1] and Tucker [2] have given necessary and sufficient conditions on a function f in order that it be in $B_1(G)$. These conditions are in terms of particular sequences of functions which converge in a uniform or a monotonic sense. Since for each ordinal $\alpha>0$, the collection $\sum_{\gamma<\alpha} B_\gamma(G)$ is a linear lattice of real functions over S and it contains the constant function over S, these results may be extended to give necessary and sufficient conditions on a function f in order that it be in $B_\alpha(G)$. In this paper we characterize the collection $B_\alpha(G)$, $\alpha>0$, in terms of an associated collection of Baire sets (Theorem 7) and give some relationships between these collections and the collections described by Hausdorff in [3].

Notation. If K is a lattice of functions, then K_u denotes the collection of all functions which are uniform limits of sequences from K, USK the collection of all functions which are limits of nonincreasing sequences from K and LSK the collection of all functions which are limits of nondecreasing sequences from K. The Baire system of functions generated by K is denoted by $B(K)$. If f is a bounded function, $\|f\|$ denotes the l.u.b. norm of f.

THEOREM 1. *If f is a bounded function in G, then f^2 is in G_u.*

Proof. Suppose f is a bounded function in G and $f \neq 0$. Let $h = f/(\|f\|)$, so that h is in G. For each n, let

$$h_n(x) = 2^n\left(\frac{i}{2^n}\right)^2\left(h(x)-\frac{i-1}{2^n}\right)-2^n\left(\frac{i-1}{2^n}\right)^2\left(h(x)-\frac{i}{2^n}\right)$$

$$= \frac{2i-1}{2^n}h(x)-\frac{i(i-1)}{2^n\cdot 2^n},$$

if $\frac{i-1}{2^n} \leqslant h(x) \leqslant \frac{i}{2^n}$, where $-2^n+1 \leqslant i \leqslant 2^n$.

If $h(x) = i\cdot 2^{-n}$, then $h_n(x) = (i\cdot 2^{-n})^2 = h^2(x)$ and if $h(x)$ is between $(i-1)\cdot 2^{-n}$ and $i\cdot 2^n$, then $h_n(x)$ is between $[(i-1)\cdot 2^{-n}]^2$ and $(i\cdot 2^{-n})^2$. So, for each x in S,

$$|h_n(x)-h^2(x)| < |(i\cdot 2^{-n})^2-[(i-1)\cdot 2^{-n}]^2| = 2^{-2n}|2i-1| \leqslant 2^{-2n}\cdot(2|i|+1).$$

So, $|h_n(x)-h^2(x)| < 2^{-2n}\cdot(2\cdot 2^n+1) = 2^{-n+1}+2^{-2n}$ and the sequence $\{h_p\}_{p=1}^{\infty}$ converges uniformly to h^2.

For each n, let $g_{ni} = \max[(i-1)\cdot 2^{-n},\ \min(h, i\cdot 2^{-n})]$, and let $c_{ni} = 2^{-n}(2i-1)$, for each i, $-2^n+1 \leqslant i < 2^n$.

Let $d_n = 1-\sum_{i=-2^n+1}^{2^n} c_{ni}\cdot i\cdot 2^{-n} = 1-\sum_{i=-2^n+1}^{2^n} i(2i-1)\cdot 2^{-n}\cdot 2^{-n}$.

Since G is a linear lattice containing the constant functions, the function $d_n+\sum_{i=-2^n+1}^{2^n} c_{ni}\cdot g_{ni}$ is in G, for each n.

Suppose n is a positive integer and $(p-1)\cdot 2^{-n} \leqslant h(x) \leqslant p\cdot 2^{-n}$, where $-2^n+1 \leqslant p \leqslant 2^n$. Then

$$d_n+\sum_{i=-2^n+1}^{2^n} c_{ni}\cdot g_{ni}(x)$$

$$= \sum_{i=-2^n+1}^{p-1}\frac{(2i-1)i}{2^n\cdot 2^n}+\frac{2p-1}{2^n}\cdot h(x)+\sum_{i=p+1}^{2^n}\frac{2i-1}{2^n}\cdot\frac{i-1}{2^n}+d_n,$$

$$= \frac{2p-1}{2^n}h(x)-\frac{p(2p-1)}{2^n\cdot 2^n}+\sum_{i=-2^n+1}^{p}\frac{(2i-1)i}{2^n\cdot 2^n}+\sum_{i=p+1}^{2^n}\frac{(2i-1)(i-1)}{2^n\cdot 2^n}+d_n,$$

$$= \frac{2p-1}{2^n}h(x)-\frac{p(2p-1)}{2^n\cdot 2^n}-\frac{1}{2^n\cdot 2^n}\cdot\sum_{i=p+1}^{2^n}(2i-1)+1.$$

But, $\sum_{i=p+1}^{2^n}(2i-1) = 2^n\cdot 2^n-p^2$. So,

$$\sum_{i=2^n+1}^{2^n} c_{ni}g_{ni}(x)+d_n = (2p-1)2^{-n}h(x)-p(p-1)\cdot 2^{-n}\cdot 2^{-n} = h_n(x).$$

This shows that for each n, h_n is in G and it follows that f^2 is in G_u.

Theorem 2. *Suppose f and g are bounded functions in G_u. Then (1) $f\cdot g$ is in G_u, (2) every polynomial in f is in G_u, and (3) if φ is a continuous real function whose domain includes the range of f and φ is the uniform limit of a sequence of polynomials, then $\varphi[f]$ is in G_u.*

Theorem 2 is a corollary to Theorem 1.

Definition. The collection to which X belongs if and only if there is some f in G and segment (a, b) such that $X = (a < f < b)$ is denoted by D.

Theorem 3. *Suppose X is a proper subset of S and X is in D. If g is the characteristic function of $S-X$, then g is in USG.*

Proof. Let f be a function in G and (a, b) a segment such that $X = (a < f < b)$. Let $h = 2\cdot(b-a)^{-1}\cdot[\max(a, \min(f, b))-(a+b)\cdot 2^{-1}]$, so that h is in G, $-1 \leqslant h \leqslant 1$ and $(-1 < h < 1) = (a < f < b) = X$. It follows from Theorem 2 that, for each n, h^{2n} is in G_u. For each n, $(h^{2n} = 1) = (f \leqslant a)+(f \geqslant b) = S-X$, and $h^{2n} \geqslant h^{2n+2}$. The sequence $\{h^{2p}\}_{p=1}^{\infty}$ is a nonincreasing sequence from G_u converging to g so that g is in $US(G_u)$. It follows that g is in USG.

Theorem 4. *In order that f be in USG it is necessary that, if a is a number, then the set $(f < a)$ should be the sum of countably many sets each belonging to D. In case f is bounded above, this condition is also sufficient.*

Proof. Suppose f is in USG. Let $\{f_p\}_{p=1}^{\infty}$ be a nonincreasing sequence from G converging to f from above. If a is a number, then $(f < a) = \sum_{n=1}^{\infty}\sum_{p=1}^{\infty}(a-n < f_p < a)$ and is the sum of countably many sets from D.

Suppose f is bounded above and for each number a, $(f < a)$ is the sum of countably many sets from D. Let $F = 1+f-\text{l.u.b.}\,f$ and let $\{r_p\}_{p=1}^{\infty}$ be a sequence of all rational numbers $\leqslant 1$.

For each i, let $(F < r_i) = \sum_{p=1}^{\infty} X_{ip}$ where, for each p, X_{ip} is in D and let $g_{ip}(x) = 1$, if x is in $S-X_{ip}$, and $g_{ip}(x) = r_i$, if x is in X_{ip}. For each n, let $f_n = \min(g_{11}, \ldots, g_{1n}, g_{21}, \ldots, g_{2n}, \ldots, g_{n1}, \ldots, g_{nn})$. The sequence $\{f_p\}_{p=1}^{\infty}$ is nonincreasing and converges to F. Let $u_{ip}(x) = 1$ if is in $S-X_{ip}$ and $i_{ip}(x) = 0$, if x is in X_{ip}. It follows from Theorem 3, that u_{ip} is in USG. So $(1-r_i)\cdot u_{ip}+r_i = g_{ip}$ is in USG. So, for each n, f_n is in USG and since $US(USG) = USG$, F is in USG. It follows that f is in USG.

THEOREM 5. *If H is a linear lattice on S containing the constant functions such that $USH \cdot LSH = H$, then $H_u = H$.*

Proof. If h is in H_u, let $\{h_p\}_{p=1}^{\infty}$ be a sequence from H such that for each p, $\|h-h_p\| \leqslant p^{-1}$. Let $a_p = \min(h_1+1, \ldots, h_p+p^{-1})$ for each p; $\{a_p\}_{p=1}^{\infty}$ is a non-decreasing sequence from H converging to h. h is in USH. Similarly, h is in LSH.

THEOREM 6. *If g is in LSG, h is in USG and $g \geqslant h$, then there is a function f in $LX \cdot USG$ such that $g \geqslant f \geqslant h$ and if $g(x) > h(x)$, then $g(x) > f(x) > h(x)$.*

Indication of Proof. Suppose $\{g_p\}_{p=1}^{\infty}$ is a nondecreasing sequence from G converging to g and $\{h_p\}_{p=1}^{\infty}$ is a nonincreasing sequence from G converging to h. For each p, let $a_p = g_p - 2^{-p}$ and $\beta_p = h_p + 2^{-p}$, so that the sequence $\{a_p\}_{p=1}^{\infty}$ is an increasing sequence from G converging to g and the sequence $\{\beta_p\}_{p=1}^{\infty}$ is a decreasing sequence from G converging to h.

Let $u_1 = a_1$ and for each n, $v_n = \max(u_n, \beta_n)$ and $u_{n+1} = \min(v_n, a_{n+1})$. It can be shown that there is a function f such that $u_1 \leqslant u_2 \leqslant u_3 \leqslant \ldots \to f \leftarrow \ldots \leqslant v_3 \leqslant v_2 \leqslant v_1$ and that $g \geqslant f \geqslant h$ and if $g(x) > h(x)$, then $g(x) > f(x) > h(x)$.

DEFINITION. If R is a collection of subsets of S, then $W_1(R)$ is the collection to which X belongs if and only if $X = \sum_{n=1}^{\infty}\big(\prod_{p=1}^{\infty} X'_{np}\big)$ where, for each n, p, $X_{n,p}$ is in R and $X'_{n,p}$ is the complement of $X_{n,p}$.

The next theorem includes a characterization of $B_1(G)$ in terms of $W_1(D)$.

THEOREM 7. *Suppose f is a function on S. Each two of the following statements are equivalent:*

(1) *f is in $B_1(G)$,*

(2) *f is in $US(LSG) \cdot LS(USG)$,*

(3) *f is the uniform limit of a sequence each term of which is the difference of two functions in USG,*

(4) *f is in $B_1(USG \cdot LSG)$, and*

(5) *for each segment (a, b), the set $(a < f < b)$ is in $W_1(D)$.*

Sierpiński has proved that statements 1 and 2 are equivalent in [1] and Tucker that statements 1 and 3 are equivalent in [2].

Proof that 2 implies 4 and 5. Suppose f is in US(LSG)$\cdot$LS(USG). Let $\{g_p\}_{p=1}^{\infty}$ be a noincreasing sequence from LSG converging to f and let $\{h_p\}_{p=1}^{\infty}$ be a non-decreasing sequence from USG converging to f. For each p, $g_p \geqslant f \geqslant h_p$ and it follows from Theorem 5 that there is a function f_p in US$G\cdot$LSG between g_p and h_p. The sequence $\{f_p\}_{p=1}^{\infty}$ converges to f. So, f is in B_1(US$G\cdot$LSG) and 2 implies 4. Suppose (a, b) is a segment.

The set $(a < f < b) = \sum_{q=1}^{\infty}\sum_{p=1}^{\infty}\left(a + \frac{b-a}{p} \leqslant h_q\right) \cdot \left(g_q \leqslant b - \frac{b-a}{p}\right) = \sum_{q=1}^{\infty}\sum_{p=1}^{\infty}\left(h_q < a + \frac{b-a}{p}\right)' \cdot \left(g_q > b - \frac{b-a}{p}\right)'$. It follows from Theorem 4 that $\left(h_q < a + \frac{b-a}{p}\right)$ and $\left(g_q > b - \frac{b-a}{p}\right)$ are the sums of countably many sets in D. So $\left(h_q < a + \frac{b-a}{p}\right)' \cdot \left(g_q > b - \frac{b-a}{p}\right)'$ is the common part of the complements of countably many sets in D and $(a < f < b)$ is in $W_1(D)$. So, statement 2 implies statement 5.

Proof that 4 implies 2. Suppose f is in B_1(US$G\cdot$LSG). Since US$G\cdot$LSG is a linear lattice and statements 1 and 2 are equivalent, f is in US[LS(US$G\cdot$LSG] and in LS[US(US$G\cdot$LSG)]. But, it is easy to show that LS(US$G\cdot$LSG) = LSG and that US(US$G\cdot$LSG) = USG. So, f is in US(LSG)$\cdot$LS(USG). Statements 2 and 4 are equivalent.

Proof that 5 implies 4. Suppose f is a function on S and for each segment (a, b), $(a < f < b)$ is in $W_1(D)$. Let $f' = 2\pi^{-1}\tan^{-1} f$. For each n, let L_n be the collection of all segments of the form $(i-1/2^n, i+1/2^n)$ where, $-2^n+1 \leqslant i \leqslant 2^n+1$. L_n is a collection of segments covering $(-1, 1)$. For each n, let $\{K_{np}\}_{p=1}^{\infty}$ be a sequence such that (1) each K_{np} is the common part of the complements of countably many sets in D, (2) for each p, there is some member (a, b) of L_n such that K_{np} is a subset of $(a < f' < b)$, and (3) for each segment (a, b) of L_n, there is a subsequence $\{K_{np_i}\}_{i=1}^{\infty}$ such that $(a < f' < b) = \sum_{i=1}^{\infty} K_{np_i}$.

For each positive integer pair n, p, let $g_{np}(x) = 1$, if x is in $S - K_{np}$ and $g_{np}(x) = b$, if x is in K_{np} and K_{np} is a subset of $(a < f' < b)$ and (a, b) is in L_n and let $h_{np}(x) = -1$, if x is in $S - K_{np}$ and $h_{np}(x) = a$, if x is in K_{np} and K_{np} is a subset of $(a < f' < b)$ and (a, b) is in L_n. For each n, p, $(h_{np} < c) = S$, if $c > a$ and $(h_{np} < c) = S - K_{np}$, if $c \leqslant a$. But $S - K_{np}$ is the sum of countably many sets in D. It follows from Theorem 4 that h_{np} is in USG. Similarly, g_{np} is in LSG.

For each n, let $a_n = \min(g_{11}, \ldots, g_{1n}, g_{21}, \ldots, g_{2n}, \ldots, g_{n1}, \ldots, g_{nn})$ and let $\beta_n = \max(h_{11}, \ldots, h_{1n}, h_{21}, \ldots, h_{2n}, \ldots, h_{n1}, \ldots, h_{nn})$. For each n, a_n is in LSG and β_n is in USG, $a_n \geqslant a_{n+1}$ and $\beta_n \leqslant \beta_{n+1}$. Noting that for each n, p, $h_{np} < f' < g_{np}$, we have that for each n, $\beta_n < f' < a_n$. It can be shown that the sequences $\{a_p\}_{p=1}^{\infty}$ and $\{\beta_p\}_{p=1}^{\infty}$ converge to f'. So, f' is in B_1(US$G\cdot$LSG). Let $\{t_p\}_{p=1}^{\infty}$ be a sequence from US$G\cdot$LSG converging to f' such that, for each n, $\|t_n\| = c_n < 1$. For each n, let $S_n = \tan \pi/2\ t_n$; it follows from Theorems 2 and 5 that S_n is in US$G\cdot$LSG. The sequence $\{S_p\}_{p=1}^{\infty}$ converges to f. f is in B_1(US$G\cdot$LSG). This completes the proof of Theorem 7.

Hausdorff in [3] defines an ordinary function system F as a collection of functions which is a linear lattice containing the constant functions such that if f and g are in F, $f\cdot g$ is in F and if there is no x such that $f(x)=0$, f^{-1} is in F. A complete ordinary function system is an ordinary system which is closed under uniform limits. In [3] Hausdorff characterized $B_1(F)$ where F is an ordinary function system in terms of associated Baire sets and proved that $B_1(F)$ is a complete ordinary function system. Theorem 7 and the following theorem strengthen these results.

THEOREM 8. *If G is a linear lattice on S containing the constant functions, then $B_1(G)$ is a complete ordinary function system.*

Proof. Certainly $B_1(G)$ is a linear lattice on S containing the constant functions. Sierpiński [1] has shown that $\mathrm{US}B_1(G)\cdot\mathrm{LS}B_1(G)=B_1(G)$. So, by Theorem 5, $B_1(G)$ is closed under uniform limits. Suppose f is in $B_1(G)$. Let $\{f_p\}_{p=1}^{\infty}$ be a sequence of bounded functions from G converging to f. According to Theorem 2, f_p^2 is in G_u for each p. It follows that f^2 is in $B_1(G)$. If g is in $B_1(G)$, since $2f\cdot g=(f+g)^2-f^2-g^2$, we have $f\cdot g$ is in $B_1(G)$. Suppose there is no x such that $f(x)=0$. Let $\{t_p\}_{p=1}^{\infty}$ be a sequence from G converging to f^2 such that for each p, t_p is bounded and $t_p(x)\geqslant p^{-1}$, for each x in S. Let $S_p=t_p^{-1}$ for each p. Again, according to Theorem 2, S_p is in the G_u. So, f^{-2} is in $B_1(G)$ and hence $f^{-1}=f\cdot f^{-2}$ is in $B_1(G)$. So, $B_1(G)$ is a complete ordinary function system.

DEFINITION. Let $W_0(D)=D$ and for each ordinal number α, $0<\alpha<\Omega$, let $W_\alpha(D)$ be $W_1\big(\sum_{\gamma<\alpha}W_\gamma(D)\big)$.

Use of transfinite induction and the fact that the collection $\sum_{\gamma<\alpha}B_\gamma(G)$ is a linear lattice containing the constant functions, yields the following theorem.

THEOREM 9. *Suppose f is a function on S and $0<\alpha<\Omega$. In order that f belong to $B_\alpha(G)$ it is necessary and sufficient that, for each segment (a,b), the set $(a<f<b)$ should be in $W_\alpha(D)$.*

Remark. In case G is a complete ordinary function system, we have the following relationships between the method presented here and the method given by Hausdorff [3, pp. 292–293]. The functions in $B_\xi(G)$ are the functions f^ξ, if $0\leqslant\xi<w$ and are the functions $f^{\xi+1}$, if $w\leqslant\xi<\Omega$. Also, the sets in $W_\xi(D)$ are the sets M^ξ, if $0\leqslant\xi<w$ and are the sets $M^{\xi+1}$ if $w\leqslant\xi<\Omega$.

THEOREM 10. *Suppose G is $USG\cdot LSG$. If for each n, A_n is in D, then $\sum_{p=1}^{\infty}A_p$ is in D.*

Proof. For each n, $A_n=(a_n<f_n<b_n)$. For each n, let $g_n=(b_n-a_n)^{-1}\big[\min\big(b_n,\max(a_n,f_n)\big)-a_n\big]$, g_n is in G and $A_n=(0<g_n<1)$.

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For each n, let $h_n=(1-g_n)\cdot g_n$. It follows from Theorems 2 and 5 that h_n is in G. Let $h=\sum_{p=1}^{\infty}2^{-p}h_p$. Again, it follows from Theorem 5 that h in G and $\sum_{p=1}^{\infty}A_p=(0<h<2)$ is in D.

Theorem 10 allows us to make a simplification in describing the collection $W_{\alpha+1}(D)$ where, $0<\alpha<\Omega$. Since $\mathrm{US}B_\alpha(G)\cdot\mathrm{LS}B_\alpha(G)=B_\alpha(G)$, it follows from Theorem 10 that the sum of countably many sets in $W_\alpha(D)$ belongs to $W_\alpha(D)$. If X is in $W_{\alpha+1}(D)$, then $X=\sum_{n=1}^{\infty}\prod_{p=1}^{\infty}X'_{pn}$. Since $\prod_{p=1}^{\infty}X'_{pn}=\big(\sum_{p=1}^{\infty}X_{pn}\big)'$ and $\mathrm{US}B_\alpha(G)\cdot\mathrm{LS}B_\alpha(G)=B_\alpha(G)$ (see Sierpiński [1], p. 13), it follows from Theorem 10, that $\sum_{p=1}^{\infty}X_{pn}$ is in $W_\alpha(D)$. So, X is in $W_{\alpha+1}(D)$ if and only if $X=\sum_{n=1}^{\infty}X_n'$ where, each X_n is in $W_\alpha(D)$.

The preceding results characterize the collection $B_\alpha(G)$ in terms of $W_\alpha(D)$ for each α, $0<\alpha<\Omega$. However, $W_0(D)=D$ may not characterize $B_0(G)=G$. For example, let G be the collection of all bounded functions over the interval $[0,1]$ which are continuous except for a countable set and let H be the collection of all function on the interval $[0,1]$ which are continuous except for a countable set. The D sets for the collection G are the D sets for the collection H. The following theorem gives some conditions under which the collection D does characterize G.

THEOREM 11. *Suppose G is $USG\cdot LSG$. The following statements are equivalent:*

(1) *If f is in G and φ is a continuous real function on the range of f, $\varphi[f]$ is in G.*

(2) *A function f is in G if and only if for each segment (a,b), $(a<f<b)$ is in D.*

Proof. By definition if f is in G, then $(a<f<b)$ is in D for each segment (a,b).

Suppose f is a function on S such that for each segment (a,b), the set $(a<f<b)$ is in D. Let $f'=2/\pi\tan^{-1}f$. For each n, let

$$g_{n,i}(x)=\begin{cases}1 \text{ if } x \text{ is in } S-\left(\dfrac{i-1}{2^n}<f'<\dfrac{i+1}{2^n}\right),\\[2ex] \dfrac{i+1}{2^n}, \text{ if } x \text{ is in } \left(\dfrac{i-1}{2^n}<f'<\dfrac{i+1}{2^n}\right),\end{cases}$$

and

$$h_{n,i}(x) = \begin{cases} -1 \text{ if } x \text{ is in } S - \left(\dfrac{i-1}{2^n} < f' < \dfrac{i+1}{2^n}\right), \\ \dfrac{i-1}{2^n}, \text{ if } x \text{ is in } \left(\dfrac{i-1}{2^n} < f' < \dfrac{i+1}{2^n}\right), \end{cases}$$

where $-2^n+1 \leqslant i \leqslant 2^n-1$. It follows from Theorem 4, that $g_{n,i}$ is in USG and $h_{n,i}$ is in LSG and we have $g_{n,i} > f' > h_{n,i}$.

Let $a_1 = \min(g_{1,-1}, g_{1,0}, g_{1,1})$ and $\beta_1 = \max(h_{1,-1}, h_{1,0}, h_{1,1})$, and for each n, $a_{n+1} = \min(g_{n,2^n-1}, g_{n,2^n-2}, \ldots, g_{n,-2^n+1}, a_1, \ldots, a_n)$, and $\beta_{n+1} = \max(h_{n,2^n-1}, h_{n,2^n-2}, \ldots, h_{n,-2^n+1}, \beta_1, \ldots, \beta_n)$. For each n, a_n is in USG and β_n is in LSG and it follows that $a_1 \geqslant a_2 \geqslant a_3 \geqslant \ldots \to f' \leftarrow \ldots \leqslant \beta_3 \leqslant \beta_2 \leqslant \beta_1$. So, f' is in US$G\cdot$LS$G = G$. By assumption, $f = \tan\left[\frac{\pi}{2} f'\right]$ is in G. So, statement 1 implies statement 2.

Now, suppose that h is in G if and only if for each segment (a, b), the set $(a < h < b)$ is in D. Suppose f is in G and φ is a continuous real function on Y_f, the range of f.

Suppose (a, b) is a segment. Let $\{(a_p, b_p)\}_{p=1}^{\infty}$ be a sequence of segments such that $(a < \varphi < b) = Y_f \cdot \sum_{p=1}^{\infty} (a_p, b_p)$ so that $(a < \varphi[f] < b) = \sum_{p=1}^{\infty} (a_p < f < b_p)$. So, the set $(a < \varphi[f] < b)$ is the sum of countably many sets in D. Since US$G\cdot$LS$G = G$, it follows from Theorem 10, that $(a < \varphi[f] < b)$ is in D. So, $\varphi[f]$ is in G. Statements 1 and 2 are equivalent.

THEOREM 12. *Suppose S is a countable set and $F(S)$ denotes the collection of all real functions on S. In order that $F(S) = B(G)$ it is necessary and sufficient that if x and y are in S, then there be a function f in G such that $f(x) \neq f(y)$. Moreover, if $F(S) = B(G)$, then $F(S) = B_1(G)$.*

Proof. Suppose that if x and y are in S, then there is some f in G such that $f(x) \neq f(y)$. Let $\{s_p\}_{p=1}^{\infty}$ be a sequence of all the points of S.

It follows that for each two positive integers n and p, there is a function t_{np} in G such that $\|t_{np}\| \leqslant 1$, $t_{np}(s_p) = 0$ and $t_{np}(s_n) = 1$. Let $t_{pp}(x) = 1$ for each p and each x in S.

Let $h_{nq} = \prod_{p=1}^{n} t_{qp}$, for each positive integer pair n, q. It follows from Theorem 2 that h_{nq} is in G_u. If $i \leqslant n$, then $h_{nq}(s_i) = 0$ if $i \neq q$ and $= 1$ if $i = q$.

Suppose f is a function on S. For each n, let $f_n = \sum_{q=1}^{n} f(s_q) h_{nq}$. If $i \leqslant n$, then $f_n(s_i) = f(s_i)$. So, f is the pointwise limit of a sequence from G_u and f is in $B_1(G)$.

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