Math 4050 Review #3

1. State the Fundamental Theorem of Algebra.
2. State Descartes's Rule of Signs.
3. State and prove the Rational Root Test.
4. State the Upper Bound and Lower Bound Rules for polynomials.
5. State and prove the Conjugate Root Theorem.
6. State and prove the Unique Factorization Theorem for polynomials over $\mathbb{C}$.
7. State the Unique Factorization Theorem for polynomials over $\mathbb{R}$.
8. Let $f(x) = (x+1)(x-2)(x+2)$, so that $f(0) = -4$ and $f(3) = 20$. Use these observations to prove that f is negative on $(-1, 2)$ and positive on $(2, \infty)$. (These are called *test points.*)
9. Prove the following rules for graphing the polynomial $f(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0$.
   - The graph is unbroken; that is, it can be drawn without lifting your pencil.
   - The graph is smooth; that is, there are no abrupt changes of direction.
   - If $f(a) < 0$ and $f(b) > 0$, then $f(x) = 0$ for at least one x between a and b.
   - The long-range behavior of $f(x)$ is the same as the long-range behavior of $y = a_n x^n$.
   - The $y-$intercept of $f(x)$ is $y = a_0$.
10. Let $f \in \mathbb{R}[x]$ have degree n. Prove that f has at most n $x-$intercepts.
11. Let $f \in \mathbb{R}[x]$ have degree n. Prove that f has at most $n-1$ local minima and maxima.
12. Let $f \in \mathbb{R}[x]$ have degree n, where n is odd. Prove that f has at least one real root.
13. Prove the Quadratic Formula.
14. Factor $f(x) = ax^2 + bx + c$ over $\mathbb{R}$ and over $\mathbb{C}$, given that $b^2 - 4ac \geq 0$.
15. Factor $f(x) = ax^2 + bx + c$ over $\mathbb{R}$ and over $\mathbb{C}$, given that $b^2 - 4ac < 0$.
16. Explain how the two formulas for compound interest can be derived for secondary students.
17. Suppose that \$3,000 is invested at 4% interest. Find the amount after 5 years if the money is compounded (a) annually, (b) monthly, (c) daily, (d) continuously.
18. Suppose that $n, m \in \mathbb{Z}$. State the most general conditions on x and y for which $(x^n)^m = x^{nm}$ and $x^n y^n = (xy)^n$, and give counterexamples for x and/or y outside of your conditions.
19. Repeat the previous problem if $n, m \in \mathbb{R}$.

20. Make hand-drawn rough sketches of the following functions, stating each domain and range:

- $f(x) = x^2$
- $f(x) = x^{3/2}$
- $f(x) = x^{-2}$
- $f(x) = x^{-3}$
- $f(x) = x^{1/2}$
- $f(x) = x^{-3/2}$
- $f(x) = x^{2/3}$
- $f(x) = x^{-7/5}$
- $f(x) = x^{\pi}$
- $f(x) = 2^x$
- $f(x) = 10^x$
- $f(x) = (0.5)^x$
- $f(x) = (0.1)^x$
- $f(x) = e^x$
- $f(x) = e^{-x}$
- $f(x) = \log_2 x$
- $f(x) = \log_{1/2} x$
- $f(x) = \ln x$

21. In the previous problem, some pairs of graphs are related to each other by a simple transformation. State which pairs, the transformation, and an algebraic justification for the relationship.

22. Prove that $f(x) = x^n$ is an odd function when n is an odd integer.

23. Assume that $(z^n)^m = z^{nm}$ for any nonzero complex number z and any $n, m \in \mathbb{Z}^+$. Carefully prove that this property remains true if $n, m \in \mathbb{Z}$.

24. Assume that $x^r y^r = (xy)^r$ for any positive real numbers x and y and any $r \in \mathbb{Z}$. Carefully prove that the property remains true if $r \in \mathbb{Q}$.

25. Find $\log_{2\sqrt{2}} \dfrac{1}{\sqrt[3]{4}}$ without a calculator.

26. Let $\alpha \in \mathbb{R}^+ \setminus \{1\}$ and let $r, s \in \mathbb{R}^+$. Assuming the Laws of Exponents, prove that $\log_\alpha(r/s) = \log_\alpha r - \log_\alpha s$.

27. Compute $\displaystyle\sum_{n=2}^{1000} \frac{1}{\log_n(1000!)}$

28. Let $f(x) = \log_\alpha x$ and $g(x) = \log_\beta x$, where $\alpha, \beta \in \mathbb{R}^+ \setminus \{1\}$.
    - Using properties of logarithms, find the constant c so that $g(x) = cf(x)$.
    - By what transformation can the graph of g be obtained from the graph of f?

29. Express $22^{2222}/5^{5555}$ in scientific notation, accurate to six significant figures.

30. For this set of problems, imagine that you are living in an era without electronic calculators. You are able to perform addition, subtraction, multiplication (and hence exponentiation if the exponent is an integer), and division. However, you cannot perform more advanced computations using only pencil and paper.
    - Explain how $\ln 10 \approx 2.302585093$ can be computed to such high precision.
    - Using the fact that $2^{10} = 1024$, calculate $\log_{10} 2$ to nine decimal places. You may use the previous result and the $+$, $-$, $\times$, and $\div$ buttons on your calcuator. You may also use the y^x button only if the exponent is an integer.
    - Calculate $\ln 2$ to nine decimal places. You may use the previous result(s) and the $+$, $-$, $\times$, and $\div$ buttons on your calcuator. You may also use the y^x button only if the exponent is an integer.

31. State the definition of an algebraic number and a transcendental number.
32.
   - Prove that every rational number is algebraic.
   - Name (without proof) a real number that is irrational but algebraic.
   - Name (without proof) two real numbers that are transcendental.
33. Prove that the sum and product of two rational numbers are both rational.
34. Prove that the sum and product of a nonzero rational number and an irrational number are both irrational.
35.
   - Exhibit two irrational numbers whose sum is irrational.
   - Exhibit two irrational numbers whose sum is rational.
   - Exhibit two irrational numbers whose product is irrational.
   - Exhibit two irrational numbers whose product is rational.
36. Prove that $\frac{5-\sqrt{3}}{2}$ and $\sqrt{3}-2\sqrt{5}$ are both irrational and algebraic.
37. Prove that i is algebraic.
38. Prove that $\log_{10} 1.5$ is irrational.