The Projective Quantization

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Introduction. Let $\operatorname{Vec}\mathbb{R}^m$ be the Lie algebra of smooth vector fields on $\mathbb{R}^m$. In this talk we will survey some results concerning the action of $\operatorname{Vec}\mathbb{R}^m$ on modules of differential operators. To begin with, let σ be the two-sided action of $\operatorname{Vec}\mathbb{R}^m$ on the associative algebra $\operatorname{Diff}\mathbb{R}^m$ of smooth differential operators on $\mathbb{R}^m$:

$$\sigma(X)T := X \circ T - T \circ X.$$

This is a derivation action which preserves the *order filtration* $\operatorname{Diff}^k\mathbb{R}^m$. The associated subquotients are the *symbol modules:*

$$\operatorname{Symb}^k\mathbb{R}^m := \operatorname{Diff}^k\mathbb{R}^m / \operatorname{Diff}^{k-1}\mathbb{R}^m.$$

We will write σ^k for the natural action of $\operatorname{Vec}\mathbb{R}^m$ on $\operatorname{Symb}^k\mathbb{R}^m$. Let $\operatorname{Symb}\mathbb{R}^m$ be the *total symbol module,* the graded algebra of $\operatorname{Diff}\mathbb{R}^m$:

$$\operatorname{Symb}\mathbb{R}^m := \bigoplus_{k=0}^{\infty} \operatorname{Symb}^k\mathbb{R}^m.$$

It is natural to ask whether the order filtration has a $\operatorname{Vec}\mathbb{R}^m$-invariant splitting. The answer is no, and the way in which subquotients $\operatorname{Symb}^k\mathbb{R}^m$ are sewn together to form $\operatorname{Diff}\mathbb{R}^m$ involves subtle combinatorics of interest in geometry, modular forms, and Lie algebra cohomology. Any linear isomorphism from $\operatorname{Symb}\mathbb{R}^m$ to $\operatorname{Diff}\mathbb{R}^m$ which respects the filtration (in the sense that it carries $\oplus_0^k \operatorname{Symb}^j\mathbb{R}^m$ to $\operatorname{Diff}^k\mathbb{R}^m$ and defines the identity on $\operatorname{Symb}^k\mathbb{R}^m$) is called a *quantization,* and its inverse is the corresponding *total symbol.* Quantizations may be used to transfer the algebra structure of $\operatorname{Diff}\mathbb{R}^m$ to $\operatorname{Symb}\mathbb{R}^m$, where it may be regarded as a non-commutative deformation of the natural commutative algebra structure of $\operatorname{Symb}\mathbb{R}^m$.

We will study the action of $\operatorname{Vec}\mathbb{R}^m$ on $\operatorname{Diff}\mathbb{R}^m$ by transferring it to $\operatorname{Symb}\mathbb{R}^m$ with a particular quantization, the unique quantization covariant with respect to the projective subalgebra. This quantization is particularly useful because the projective subalgebra is maximal. During the course of the analysis one is led to consider differential operators between arbitrary *tensor density modules.* So far this program has been completed only for $m = 1$ [CMZ97], but there are partial results for general m [LO99].

The Projective Subalgebra. Write D_i for $\partial/\partial x_i$ and let E denote the *Euler operator* $\sum_{i=1}^m x_i D_i$. The *projective subalgebra* $\mathfrak{a}_m$ of $\operatorname{Vec}\mathbb{R}^m$ is

$$\mathfrak{a}_m := \operatorname{Span}_{\mathbb{C}}\big\{D_i, x_j D_i, x_j E : 1 \le i, j \le m\big\}.$$

It is isomorphic to $\mathfrak{sl}_{m+1}$.

Proposition 1 *There exists a unique quantization*

$$\mathrm{PQ} : \mathrm{Symb}\,\mathbb{R}^m \to \mathrm{Diff}\,\mathbb{R}^m$$

which intertwines the $\mathfrak{a}_m$*-actions* $\oplus_{k=0}^{\infty}\sigma^k|_{\mathfrak{a}_m}$ *and* $\sigma|_{\mathfrak{a}_m}$*, the* projective quantization. *Its inverse* PS *is called the* projective total symbol.

Idea of Proof. Since $\mathfrak{a}_m \cong \mathfrak{sl}_{m+1}$, the machinery of semisimple Lie algebras may be brought to bear. In particular one has the Casimir operator of $\mathfrak{a}_m$, which acts on any $\mathfrak{a}_m$-module so as to commute with the action of $\mathfrak{a}_m$. In this case one finds easily that it acts by distinct scalars on the subquotients $\mathrm{Symb}^k\,\mathbb{R}^m$ (in fact, under $\mathfrak{a}_m$ the $\mathrm{Symb}^k\,\mathbb{R}^m$ are duals of $\mathfrak{gl}_m$-relative Verma modules). Thus $\mathrm{Diff}\,\mathbb{R}^m$ splits into its eigenspaces, proving the theorem. □

Remark. The projective quantization is given by a rather complicated formula, computed explicitly for $m = 1$ in [CMZ97] and for all m in [LO99]. One might ask, why not use a simpler quantization invariant under a smaller subalgebra? For example, the naïve quantization

$$\mathrm{Symb}^k\,\mathbb{R}^m \to \mathrm{Span}_{C^\infty\mathbb{R}^m}\big\{D_1^{i_1}\cdots D_m^{i_m} : \sum_r i_r = k\big\}$$

is invariant under the *affine subalgebra* $\mathrm{Span}_{\mathbb{C}}\big\{D_i, x_jD_i : 1 \le i,j \le m\big\}$ (a maximal parabolic subalgebra of $\mathfrak{a}_m$). The answer is that the benefits of the covariance of PQ under the larger subalgebra $\mathfrak{a}_m$ outweigh the difficulties caused by the complexity of its formula. The fact that the projective quantization is unique means that many difficult questions about $\mathrm{Diff}\,\mathbb{R}^m$ become easy once they are transferred to $\mathrm{Symb}\,\mathbb{R}^m$ via PQ. We now define the transferred $\mathrm{Vec}\,\mathbb{R}^m$-action and give a lemma stating its most elementary properties.

Definition. *Let* π *be the action* $\sigma^{\mathrm{PS}} := \mathrm{PS}\circ\sigma\circ\mathrm{PQ}$ *of* $\mathrm{Vec}\,\mathbb{R}^m$ *on* $\mathrm{Symb}\,\mathbb{R}^m$*. We will regard* π *as a matrix with entries*

$$\pi_{ij} : \mathrm{Vec}\,\mathbb{R}^m \to \mathrm{Hom}_{\mathbb{C}}(\mathrm{Symb}^j\,\mathbb{R}^m, \mathrm{Symb}^i\,\mathbb{R}^m).$$

Lemma 2 *(a) The matrix* π *is upper triangular.*

(b) Its diagonal entries are $\pi_{ii} = \sigma^i$*, the natural actions on the* $\mathrm{Symb}^i\,\mathbb{R}^m$*.*

(c) The entries π_{ij} *above the diagonal are zero on* $\mathfrak{a}_m$ *and are* $\mathfrak{a}_m$*-covariant maps from* $\mathrm{Vec}\,\mathbb{R}^m$ *to* $\mathrm{Hom}_{\mathbb{C}}(\mathrm{Symb}^j\,\mathbb{R}^m, \mathrm{Symb}^i\,\mathbb{R}^m)$*.*

Idea of Proof. Part (a) is due to the fact that σ and PQ preserve the filtration. Part (b) holds because PQ induces the identity map on $\mathrm{Symb}^i\,\mathbb{R}^m$. Part (c) follows from the $\mathfrak{a}_m$-covariance of PQ. □

The goal is to compute the matrix entries π_{ij}. Let us give a simple example of the kind of data they contain: the subquotient $\mathrm{Diff}^k\,\mathbb{R}^m/\,\mathrm{Diff}^l\,\mathbb{R}^m$ splits as $\oplus_{l<i\le k}\,\mathrm{Symb}^i\,\mathbb{R}^m$ under the entire vector field Lie algebra $\mathrm{Vec}\,\mathbb{R}^m$ if and only if $\pi_{ij} = 0$ for $l < i < j \le k$. (The "if" statement would hold with any quantization replacing PQ, but the "only if" statement holds only for PQ.)

Cohomology. Without defining Lie algebra cohomology, let us briefly state the cohomological properties of the π_{ij}. Lemma 2c says that the upper triangular entries are $\mathfrak{a}_m$-relative 1-cochains. The fact that π is a representation translates to the *cup equation:*

$$\partial\pi_{ij} + \sum_{i<r<j} \pi_{ir} \cup \pi_{rj} \;=\; 0, \tag{1}$$

where ∂ is the coboundary operator. In particular, the entries $\pi_{i,i+1}$ on the first superdiagonal are 1-cocycles. The uniqueness of the projective quantization implies that they are cohomologically trivial if and only if they are zero.

The Case $m > 1$. Here Lecomte and Ovsienko have computed the entries $\pi_{i,i+1}$ and $\pi_{i,i+2}$ on the first and second superdiagonals [LO99]. They also proved that the entries on the higher superdiagonals are determined up to 2 parameters by their $\mathfrak{a}_m$-relativity alone, but to our knowledge they have not yet been computed.

For $m > 1$, the cocycles $\pi_{i,i+1}$ on the first superdiagonal are in general non-trivial. However, their cup products $\pi_{i,i+1} \cup \pi_{i+1,i+2}$ are always zero [AALO02], so the second superdiagonal entries $\pi_{i,i+2}$ are also cocycles. They are non-trivial, and none of the higher superdiagonal entries is a cocycle [LO00].

The Case $m = 1$. Here Cohen, Manin, and Zagier have computed all of the π_{ij} in connection with their work on modular forms [CMZ97]. Those on the first superdiagonal are zero, and so by the cup equation those on the second and third are cocycles (those on the second are closely related to the Gel'fand-Fuks cocycle, which occurs in the coadjoint representation of the Virasoro Lie algebra). All cup products $\pi_{i,i+2} \cup \pi_{i+2,i+4}$ between the second superdiagonal entries are zero (a consequence of the fact that multiplication in $\operatorname{Diff}\mathbb{R}^m$ is commutative up to symbol), so the fourth superdiagonal entries $\pi_{i,i+4}$ are also cocycles.

It turns out that the π_{ij} are differential operator-valued for all m. In order to explain this for $m = 1$, we must define the *tensor density module of degree* λ of $\operatorname{Vec}\mathbb{R}$:

$$F_\lambda := \left\{ dx^\lambda f(x) : f \in C^\infty\mathbb{R} \right\}.$$

The action π_λ of $\operatorname{Vec}\mathbb{R}$ on F_λ is

$$\pi_\lambda(gD)(dx^\lambda f) := dx^\lambda(gf' + \lambda g' f).$$

Algebraically, the dx^λ is just a "place holder", but it has geometric meaning. For example, F_0 is the functions, F_1 is the densities, and F_{-1} is the adjoint representation of $\operatorname{Vec}\mathbb{R}$. The following lemma is an easy exercise.

Lemma 3 *The* $\operatorname{Vec}\mathbb{R}$ *modules* $(\sigma^k, \operatorname{Symb}^k\mathbb{R})$ *and* (π_{-k}, F_{-k}) *are equivalent.*

Thus the entries π_{ij} may be viewed as maps from $\operatorname{Vec}\mathbb{R}$ to $\operatorname{Hom}(F_{-j}, F_{-i})$. As such, they are differential operators, and in order to compute them it is necessary to generalize all the preceding analysis to the space

$$\operatorname{Diff}_{\lambda,p}\mathbb{R} := \big\{\text{Differential Operators} : F_\lambda \to F_{\lambda+p}\big\}.$$

This space is defined in the natural way, and it carries a two-sided action $\sigma_{\lambda,p}$ of $\operatorname{Vec}\mathbb{R}$:

$$\sigma_{\lambda,p}(X)T := \pi_{\lambda+p}(X) \circ T - T \circ \pi_\lambda(X).$$

This action preserves the order filtration $\operatorname{Diff}^k_{\lambda,p}\mathbb{R}$, and so one has the symbol modules

$$\operatorname{Symb}^k_{\lambda,p}\mathbb{R} := \operatorname{Diff}^k_{\lambda,p}\mathbb{R}/\operatorname{Diff}^{k-1}_{\lambda,p}\mathbb{R}.$$

Let $\operatorname{Symb}_{\lambda,p}\mathbb{R}$ be the total symbol module $\oplus_k \operatorname{Symb}^k_{\lambda,p}\mathbb{R}$.

In this setting, a quantization is a linear isomorphism from $\operatorname{Symb}_{\lambda,p}\mathbb{R}$ to $\operatorname{Diff}_{\lambda,p}$ which preserves the filtration. The following lemma gives necessary and sufficient conditions under which there is a unique projective quantization. (Projective quantizations exist in certain other special cases, but they are not unique.)

Proposition 4 *Let us say that p is* regular *if it is not one of* $1, 3/2, 2, 5/2, \ldots$. *There exists a unique projective (i.e., $\mathfrak{a}_1$-covariant) quantization*

$$\operatorname{PQ}_{\lambda,p} : \operatorname{Symb}_{\lambda,p}\mathbb{R} \to \operatorname{Diff}_{\lambda,p}\mathbb{R}$$

if and only if p is regular. We write $\operatorname{PS}_{\lambda,p}$ *for its inverse.*

Idea of Proof. Lemma 3 has the following generalization: $\operatorname{Symb}^k_{\lambda,p}\mathbb{R}$ is $\operatorname{Vec}\mathbb{R}$-equivalent to F_{p-k}. Therefore

$$\operatorname{Symb}_{\lambda,p}\mathbb{R} \overset{\operatorname{Vec}\mathbb{R}}{\cong} \bigoplus_{k=0}^{\infty} F(p-k). \tag{2}$$

Now the Casimir operator has eigenvalue $\lambda^2 - \lambda$ on $F(\lambda)$, so $F(1-\lambda)$ is the only other tensor density module with the same eigenvalue. From here the proof proceeds as in Proposition 1: for p regular there are no repeated eigenvalues. □

The Regular Case. In order to analyze $\operatorname{Diff}_{\lambda,p}\mathbb{R}$ we must consider the representation

$$\pi^{\lambda,p} := \sigma^{\operatorname{PS}_{\lambda,p}}_{\lambda,p} := \operatorname{PS}_{\lambda,p} \circ \sigma_{\lambda,p} \circ \operatorname{PQ}_{\lambda,p}$$

of $\operatorname{Vec}\mathbb{R}$ on $\operatorname{Symb}_{\lambda,p}$, defined for all regular p (note that $\pi^{0,0}$ is the representation π considered before). Let us use (2) to regard $\pi^{\lambda,p}$ as a representation on $\oplus_{k=0}^{\infty} F_{p-k}$. Then it becomes a matrix with entries

$$\pi^{\lambda,p}_{ij} : \operatorname{Vec}\mathbb{R} \to \operatorname{Hom}_{\mathbb{C}}(F_{p-j}, F_{p-i}).$$

Formulas for these entries are implicit in [CMZ97] and explicit in [Con01]. Like the π_{ij}, they are differential operator-valued. It can be shown that $\pi_{ij}^{\lambda,p}$ maps $\operatorname{Vec}\mathbb{R}$ into $\operatorname{Diff}_{p-j,j-i}^{j-i-2}\mathbb{R}$ and that its image is $\mathrm{PS}_{p-j,j-i}(F_2)$.

In fact, [CMZ97] contains much more than the formula for the $\pi_{ij}^{\lambda,p}$: it gives the formula for composition of differential operators between arbitrary tensor density modules in terms of the projective quantization in all regular cases. To explain, continue to use the identification (2). Then the map

$$\text{Composition} : \operatorname{Diff}_{\lambda+p,q}\mathbb{R} \otimes \operatorname{Diff}_{\lambda,p}\mathbb{R} \to \operatorname{Diff}_{\lambda,p+q}\mathbb{R}$$

may be transferred via $\mathrm{PQ}_{\lambda+p,q}$, $\mathrm{PQ}_{\lambda,p}$, and $\mathrm{PQ}_{\lambda,p+q}$ to an $\mathfrak{a}_1$-map

$$\text{Composition}^{\mathrm{PS}} : \Big(\bigoplus_{i=0}^{\infty} F_{q-i}\Big) \otimes \Big(\bigoplus_{j=0}^{\infty} F_{p-j}\Big) \to \bigoplus_{k=0}^{\infty} F_{p+q-k}.$$

The main result of [CMZ97] is the formula for $\text{Composition}^{\mathrm{PS}}$ for all non-singular values of p, q, and $p+q$. It is given in terms of *transvectants* [Gor1887], which are essentially the same as *Rankin-Cohen brackets* [Ra56, Coh75]. The transvectant $J^r_{\lambda,\nu}$ is, up to a scalar, the unique $\mathfrak{a}_1$-covariant map from $F_\lambda \otimes F_\nu$ to $F_{\lambda+\nu+r}$ (disregarding certain special cases). One begins with the observation that the component of $\text{Composition}^{\mathrm{PS}}$ mapping $F_{q-i} \otimes F_{p-j}$ to F_{p+q-k} must be a multiple of the transvectant $J^{i+j-k}_{q-i,p-j}$. The calculation of the multiple is quite difficult and leads to interesting combinatorics.

The Singular Case. When p is one of 1, 3/2, 2, 5/2, ..., some of the eigenvalues of the Casimir operator on the symbol module $\oplus_k F_{p-k}$ are doubled. This case is important in classical projective geometry; see for example [Wil1906] and [Bo49]. There is still a projective quantization

$$\mathrm{PQ}_{\lambda,p} : \bigoplus_{k=0}^{\infty} F_{p-k} \to \operatorname{Diff}_{\lambda,p}\mathbb{R},$$

but in order to make it an $\mathfrak{a}_1$-equivalence one must in general tie together into a single indecomposable $\mathfrak{a}_1$-module each pair of tensor density modules F_{p-i} and F_{p-j} with the same eigenvalue (this happens when $i+j = 1-2p$). The resulting representation $\pi^{\lambda,p} = \sigma_{\lambda,p}^{\mathrm{PS}_{\lambda,p}}$ was analyzed in [Ga00] and [CS04].

In certain special singular cases no indecomposable $\mathfrak{a}_1$-modules arise, so one has a projective quantization in the regular sense, albeit not a unique one. This occurs for example when p is a positive integer and conjugation of differential operators defines an involution of $\operatorname{Diff}_{\lambda,p}\mathbb{R}$. Since conjugation is a $\operatorname{Vec}\mathbb{R}$-equivalence from $\operatorname{Diff}_{\lambda,p}\mathbb{R}$ to $\operatorname{Diff}_{1-p-\lambda,p}\mathbb{R}$, these cases have $2\lambda = 1-p$. The space $\operatorname{Diff}_{-1/2,2}\mathbb{R}$ is a well-known example: it arises in Sturm-Liouville theory because $\operatorname{Diff}^2_{-1/2,2}\mathbb{R}$ has a $\operatorname{Vec}\mathbb{R}$-submodule equivalent to the coadjoint representation of the Virasoro Lie algebra.

The Case $m > 1$ Again. Here too there are tensor density modules: F_λ becomes $\{dx^\lambda f : f \in C^\infty \mathbb{R}^m\}$, and elements of $\operatorname{Vec}\mathbb{R}^m$ act by

$$\pi_\lambda(X)(dx^\lambda f) := dx^\lambda\big(X(f) + \lambda(\nabla \cdot X)f\big).$$

Defining $\operatorname{Diff}_{\lambda,p}\mathbb{R}^m$ as for $m = 1$, one finds again that for generic values of p the eigenvalues of the Casimir operator of $\mathfrak{a}_m$ prove the existence of a projective quantization $\mathrm{PQ}_{\lambda,p}$, which can be used to transfer the $\operatorname{Vec}\mathbb{R}^m$-action on $\operatorname{Diff}_{\lambda,p}\mathbb{R}^m$ to an action $\pi^{\lambda,p}$ on $\operatorname{Symb}_{\lambda,p}\mathbb{R}^m$ with matrix entries $\pi^{\lambda,p}_{ij}$.

The paper [LO99] actually carries out the analysis described in our earlier discussion of the $m > 1$ case for all $\pi^{\lambda,0}$, not only for $\pi = \pi^{0,0}$. To our knowledge nothing is yet known about the other $\pi^{\lambda,p}$, but rough calculation shows that in the regular cases their matrix entries are still determined up to 2 parameters by their $\mathfrak{a}_m$-relativity.

One of the goals in [LO99], [LO00], and several papers listed in their references is to classify the $\operatorname{Vec}\mathbb{R}^m$-equivalences between the subquotient modules $\operatorname{Diff}^k_{\lambda,p}\mathbb{R}^m/\operatorname{Diff}^l_{\lambda,p}\mathbb{R}^m$. This program is completed for all $m = 1$ cases, and for those $m > 1$ cases with $p = 0$.

An important new phenomenon for $m > 1$ is that the kth symbol module $\operatorname{Symb}^k_{\lambda,p}\mathbb{R}^m$ is not a tensor density module for $k > 0$. It seems desirable to enlarge the set of tensor density modules to some set of *generalized tensor density modules*, chosen so that the symbol modules of the differential operators between generalized tensor density modules are again generalized tensor density modules. The minimal such enlargement is as follows. Let L_0 be the subalgebra of $\operatorname{Vec}\mathbb{R}^m$ of vector fields vanishing at 0, and let L_1 be the subalgebra of L_0 of vector fields vanishing to second order at 0. Then L_1 is an ideal in L_0 and $L_0 = \mathfrak{gl}_m \oplus L_1$, where $\mathfrak{gl}_m$ is the Levi subalgebra $\{x_j D_i : 1 \le i, j < m\}$ of $\mathfrak{a}_m$.

Given any finite dimensional representation λ of $\mathfrak{gl}_m$, define the generalized tensor density module F_λ of $\operatorname{Vec}\mathbb{R}^m$ to be the dual of $\operatorname{Ind}_{L_0}^{\operatorname{Vec}\mathbb{R}^m}\lambda$, where λ is extended to a representation of L_0 trivial on L_1. These modules have the desired closure properties under passage to symbols of differential operators, and the ordinary tensor density modules are obtained by taking λ 1-dimensional.

It would be interesting to extend the program for $m = 1$ described above to the generalized F_λ. It will not be difficult to prove the existence of the projective quantization in the regular cases by replacing the eigenvalues of the Casimir operator with infinitesimal characters of $\mathfrak{a}_m$, but the calculation of the matrix entries will be founded on precise data on decompositions of tensor products of $\mathfrak{a}_m$-modules and is probably hard.

Cohomology Again. In the $m = 1$ case, Goncharova [Gon73] has computed the formal $\operatorname{Vec}\mathbb{R}$-cohomology of the tensor density modules, and Feigin and Fuks [FF80] have computed the formal $\operatorname{Vec}\mathbb{R}$-cohomology of the modules $\operatorname{Diff}_{\lambda,p}\mathbb{R}$. Composition of differential operators defines a cup product on the cohomology of $\oplus_{\lambda,p}\operatorname{Diff}_{\lambda,p}\mathbb{R}$. It follows from [Gon73] that this cup product is trivial at the

symbol level, but it has not yet been computed (except on H^1: see [Con01]). For $m > 1$, the only results we know of are the calculation of the 1-cohomology of the differential operators between the symbol bundles of $\mathrm{Diff}_{\lambda,0}\,\mathbb{R}^m$ [LO00], and certain of their cup products [AALO02].

It was proven in [FF80] that the $\mathfrak{a}_1$-relative 1-cohomology space of $\mathrm{Diff}_{\lambda,p}\,\mathbb{R}$ is 1-dimensional for all $\lambda \neq (1-p)/2$ when $p = 2$, 3, or 4, and also for the special values $2\lambda = -5 \pm \sqrt{19}$ when $p = 6$. It is zero for all other values of (λ, p), with the exception of some "fake" cases at $p = 1$ and $p = 5$ obtained by composing the $\mathrm{Vec}\,\mathbb{R}$-covariant map $dxD : F_0 \to F_1$ with a $p = 2$ or $p = 4$ case.

Recall that in the regular $m = 1$ case, the matrix entries $\pi_{ij}^{\lambda,p}$ are $\mathrm{Diff}_{p-j,j-i}\,\mathbb{R}$-valued 1-cocycles for $j - i = 2, 3, 4$. Thus the $\mathfrak{a}_1$-relative 1-cohomology classes with $p = 2, 3, 4$ discovered in [FF80] occur on the second, third, and fourth superdiagonals of the representations $\pi^{\lambda,q}$. In general, $\pi_{ij}^{\lambda,q}$ is not a cocycle for $j - i > 4$, but it is natural to ask whether the special $p = 6$ cocycle occurs on the sixth superdiagonal of $\pi^{\lambda,q}$ for any special values of λ, q, and j (we first heard this question posed by J. Germoni and O. Mathieu). In fact, provided that one admits pseudodifferential operators, it is not hard to use [CMZ97] to show that it does [Con05].

Indecomposable Representations. It is well-known that for any two irreducible representations V_1 and V_2 of a Lie algebra $\mathfrak{g}$, the $\mathrm{Hom}(V_1, V_2)$-valued 1-cohomology group of $\mathfrak{g}$ classifies the indecomposable representations with composition series (V_1, V_2). For example, one of the corollaries in [FF80] is that there is an $\mathfrak{a}_1$-split indecomposable representation of $\mathrm{Vec}\,\mathbb{R}$ with composition series $(F_\lambda, F_{\lambda+p})$ for all $\lambda \neq (1-p)/2$ if $p = 2$, 3, or 4, for $2\lambda = -5 \pm \sqrt{19}$ if $p = 6$, and otherwise only in the fake cases at $p = 1$ and 5.

Germoni has shown that the indecomposable representations of $\mathrm{Vec}\,\mathbb{R}$ cannot be classified [Ge01]. However, it may be possible to classify the *uniserial representations.* A representation with composition series $(V_1, \ldots, V_n)$ is said to be uniserial (or completely indecomposable) of length n if all of the V_i are irreducible and all subquotients of any length are indecomposable. In light of the cup equation (1), the question of existence of uniserial representations with a prescribed composition series involves the computation of cup products of $\mathrm{Hom}(V_i, V_{i+1})$-valued 1-cocycles. The $\mathfrak{a}_1$-split uniserial representations of $\mathrm{Vec}\,\mathbb{R}$ of length 3 with composition series $(F_\lambda, F_{\lambda+p_1}, F_{\lambda+p_1+p_2})$ were classified in [Con01]: they exist for most λ if p_1 and p_2 are 2, 3, or 4 and $p_1 + p_2 < 7$, and in a few other special cases, the most interesting of which is $p_1 = p_2 = 4$ and $2\lambda = -7 \pm \sqrt{39}$.

Let us refer to uniserial $\mathrm{Vec}\,\mathbb{R}$-modules of length $k+1$ with composition series $(F_\lambda, F_{\lambda+p_1}, \ldots, F_{\lambda+p_1+\cdots+p_k})$ as modules "of jump $(p_1, \ldots, p_k)$ at λ". Germoni suggested to me that such modules might be realized as subquotients of the modules $\mathrm{Diff}_{\lambda,p}\,\mathbb{R}$ and their pseudodifferential operator analogs. Following this idea using the results of [CMZ97], [Ga00], and [CS04], one discovers numerous examples, both $\mathfrak{a}_1$-split and otherwise [Con05]. In particular, the jump 6 mod-

ules at $2\lambda = -5 \pm \sqrt{19}$ and the jump $(4,4)$ modules at $2\lambda = -7 \pm \sqrt{39}$ are both realized in this way. They turn out be the first two terms in a sequence of pseudodifferential operator subquotients: there is a

$$\text{jump } (4,2,2,\ldots,2,4) \text{ subquotient of length } k \text{ at } 2\lambda = -2k-1 \pm \sqrt{4k^2+3}$$

for all k (in length 2 this is the jump 6).

We conclude by mentioning two of the most amusing cases: there exists a

$$\text{jump } (3,3,2) \text{ subquotient at } 8\lambda = -27 \pm \sqrt{649},$$

and a

$$\text{jump } (3,2,2,2) \text{ subquotient at } 16\lambda = -67 \pm \sqrt{3529}.$$

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