Representations of finite length composed of tensor density representations on the circle

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Let $\mathrm{Vec}(S^1)$ be the Lie algebra of polynomial vector fields on the circle, with basis

$$\{e_n = z^{n+1}\tfrac{d}{dz} : n \in \mathbb{Z}\}.$$

In this talk we will discuss indecomposable representations of $\mathrm{Vec}(S^1)$ composed of two or three tensor density representations. Let us begin by mentioning some references and related work. The length 2 case was treated by Martin and Piard [MP91], and the length 3 case was treated in [Co01]. The analogous length 2 case for $\mathrm{Vec}(\mathbb{R})$ was treated by Feigin and Fuks [FF80]. Indecomposable representations are built up from their constituent irreducible representations using certain 1-cocycles and 1-cochains, which in our setting turn out to take values in the spaces of differential operators between tensor density representations. Recently, Lecomte, Ovsienko, Roger, and several others have considered algebras of such differential operators, not only for $\mathrm{Vec}(S^1)$ but also for $\mathrm{Vec}(\mathbb{R}^n)$ [BO98, GO96, LO00, Ov97, OR98]. Simultaneously, Cohen, Manin and Zagier obtained powerful results on these differential operators under the action of the projective subalgebra of $\mathrm{Vec}(S^1)$, the infinitesimal linear fractional transformations [CMZ97].

For our purposes, tensor density representations (TDRs) are formally defined algebraic objects: for parameters $a, \gamma \in \mathbb{C}$, the TDR $A(a, \gamma)$ is the representation of $\mathrm{Vec}(S^1)$ with basis

$$\{(dz)^\gamma z^{\lambda-\gamma} : \lambda \in a + \mathbb{Z}\}$$

and action $e_n(dz^\gamma z^{\lambda-\gamma}) = (\lambda + n\gamma)dz^\gamma z^{n+\lambda-\gamma}$. Note that e_0 acts with spectrum $a + \mathbb{Z}$, so a is only determined up to a coset of $\mathbb{Z}$. One checks that the TDRs are irreducible except for $A(0,0)$ and $A(0,1)$: the constant functions $\mathbb{C}z^0$ form a trivial subrepresentation of $A(0,0)$ and the quotient $\tilde{A} = A(0,0)/\mathbb{C}z^0$ is irreducible, and $A(0,1)$ is the dual of $A(0,0)$.

Verifying a conjecture of Kac [Ka82], it was proven by Mathieu [Ma92] (see also the paper [MP91] of Martin and Piard) that any irreducible admissible representation of either $\mathrm{Vec}(S^1)$ or its universal central extension, the Virasoro Lie algebra, is a TDR, $\tilde{A}$, the trivial representation, or a quotient of a Verma module. Here *admissible* means that e_0 acts semisimply with finite dimensional weight spaces.

In this talk it will suffice to consider only those TDRs with $a = 0$, and so we will write $A(\gamma)$ for $A(0, \gamma)$ and π_γ for the action of $\mathrm{Vec}(S^1)$ on it. We will ignore the fact that $A(0)$ and $A(1)$ are reducible; the resulting inaccuracies are easily corrected.

Recall the definition of representations of finite length in this context: a representation of length n with composition series $\{A(\gamma_1), \ldots, A(\gamma_n)\}$ is a representation

π of $\mathrm{Vec}(S^1)$ on the vector space $\oplus_1^n A(\gamma_i)$, such that as a block matrix with entries

$$\pi_{ij} : \mathrm{Vec}(S^1) \to \mathrm{Hom}[A(\gamma_j), A(\gamma_i)],$$

π is lower triangular with the TDRs $\pi_{ii} = \pi_{\gamma_i}$ on the diagonal. Regarding the π_{ij} as 1-cochains, it is well-known that the condition that π be a representation is equivalent to the system of equations

$$\partial \pi_{ij} + \sum_{i>m>j} \pi_{im} \cup \pi_{mj} = 0 \quad \forall \quad i > j,$$

where the cup product is defined via composition.

The Length 2 Case. Suppose that π is a representation of $\mathrm{Vec}(S^1)$ with composition series $\{A(\gamma), A(\gamma + p)\}$. Then π_{21} is a $\mathrm{Vec}(S^1)$-cocycle with values in $\mathrm{Hom}[A(\gamma), A(\gamma+p)]$, and equivalences alter it by coboundaries and non-zero scalars. It follows that here the equivalence classes of indecomposable representations are in bijection with $\mathbb{P}H^1(\mathrm{Vec}(S^1), \mathrm{Hom}[A(\gamma), A(\gamma + p)])$. This 1-cohomology space was calculated by Martin and Piard:

Theorem 1 [MP92]. *The space $H^1(\mathrm{Vec}(S^1), \mathrm{Hom}[A(\gamma), A(\gamma+p)])$ is 2-dimensional if $p = 0$, 1-dimensional if $p = 2$, 3, or 4, and 1-dimensional for special values of γ if $p = 5$ or 6 ($\gamma = -4$ or 0 if $p = 5$ and $\gamma = \frac{1}{2}(-5 \pm \sqrt{19})$ if $p = 6$). Otherwise it is zero, excepting special cases arising from the existence of a $\mathrm{Vec}(S^1)$-intertwining map from $A(0)$ to $A(1)$.*

We make two remarks. First, Feigin and Fuks made the analogous calculation for $\mathrm{Vec}(\mathbb{R})$ [FF80]. The result is the same but the proof for $\mathrm{Vec}(S^1)$ requires additional computation. Second, the case $p = 2$ is related to the Schwarzian derivative, and Bouarroudj and Ovsienko have generalized this derivative by globalizing the cases at $p = 3, 4, 5$ to the group of diffeomorphisms of the circle [BO98]. They also proved that the cases at $p = 6$ do not globalize.

The Length 3 Case. Suppose now that π is a representation of $\mathrm{Vec}(S^1)$ with composition series $\{A(\gamma), A(\gamma + p), A(\gamma + p + q)\}$. Here the subdiagonal matrix entries π_{32} and π_{21} are cocycles and π_{31} must satisfy $\partial \pi_{31} = -\pi_{32} \cup \pi_{21}$. Results are known only in the *regular* case, in which the projective subalgebra

$$\mathfrak{a} = \mathrm{Span}\{e_{-1}, e_0, e_1\} \cong \mathfrak{sl}(2)$$

of $\mathrm{Vec}(S^1)$ acts semisimply. We remark that when $\mathrm{Vec}(S^1)$ is replaced by $\mathrm{Vec}(\mathbb{R}^n)$, $\mathfrak{a}$ becomes a copy of $\mathfrak{sl}(n + 1)$. This setting is considered in [LO00].

The Casimir operator of $\mathfrak{a}$ is $Q = e_0^2 + e_0 - e_{-1}e_1$, and one checks that it acts on $A(\gamma)$ by the scalar $\gamma(\gamma - 1)$. We define the regular case by the condition that the scalars $\gamma(\gamma - 1)$, $(\gamma + p)(\gamma + p - 1)$, and $(\gamma + p + q)(\gamma + p + q - 1)$ be distinct. In this case it is standard that up to equivalence, π_{32}, π_{21}, and π_{31} are $\mathfrak{a}$-relative 1-cochains.

The interesting cases are those in which π_{32} and π_{21} are non-trivial 1-cocycles, the *completely indecomposable* cases. Here regularity and Theorem 1 together imply that p and q are 2, 3, 4, 5, or 6, and we obtain the following result:

Theorem 2 [Co01]. *Regular completely indecomposable representations with composition series* $\{A(\gamma), A(\gamma+p), A(\gamma+p+q)\}$ *occur only for* $p+q = 4$, 5, *or* 6 *when* γ *is not equal to certain special values, and for* $p+q = 7$ *or* 8 *when* γ *is equal to certain special values. When* $p+q = 4$ *there is a 1-parameter family of such representations, and in the other cases there is only one such representation.*

Outlines of Proofs. We begin with Theorem 1 in the regular case that $\gamma(\gamma-1) \neq (\gamma+p)(\gamma+p-1)$, *i.e.,* $p \neq 0$ or $1-2\gamma$. Here the 1-cocycle π_{21} may be taken to be $\mathfrak{a}$-relative, and we will assume that it is non-trivial.

The $\mathfrak{a}$-relativity implies that $\pi_{21}(e_{\pm 2})$ are annihilated by $e_{\mp 1}$, which determines them up to scalars. It also implies that $\pi_{21}(e_2)$ determines $\pi_{21}(e_n)$ for $n > 2$, and similarly $\pi_{21}(e_{-2})$ determines $\pi_{21}(e_n)$ for $n < -2$, so there are only two degrees of freedom for π_{21}. The equation $\partial\pi_{21}(e_{-2} \wedge e_2) = 0$ removes one of them, and also (after a long calculation) implies that $p \in 1 + \mathbb{N}$. The case $p = 1$ arises from the intertwining map $A(0) \to A(1)$ and so we will discount it. The end result is that π_{21} can exist only for $p \in 2 + \mathbb{N}$, and in this case the only candidate for it (up to a scalar) is a certain special $\mathfrak{a}$-relative 1-cochain which we denote by $\beta_p(\gamma)$.

It remains only to determine those p and γ such that $\partial\beta_p(\gamma) = 0$, which leads to the theorem. It turns out that $\beta_p(\gamma)(e_n)$ is a differential operator of degree $p-2$, and the calculation is best done using the work [CMZ97] of Cohen, Manin, and Zagier in which splittings of spaces of differential operators under the action of $\mathfrak{a}$ are studied.

It is helpful to use the fact that since $\partial\beta_p(\gamma)$ is $\mathfrak{a}$-relative, we need only check that it is zero on a set of generators of $\Lambda^2(\mathrm{Vec}(S^1)/\mathfrak{a})$ under $\mathfrak{a}$. Using symmetry under the Cartan involution, this comes down to checking that $\partial\beta_p(\gamma)(l_\mu) = 0$ for $\mu = 0, 5, 7, 9, \ldots$, where $l_0 = e_{-2} \wedge e_2$ and for $\mu = 5, 7, \ldots$, l_μ is the unique $\mathfrak{a}$-lowest weight vector in $\Lambda^2(\mathrm{Vec}(S^1)/\mathfrak{a})$ of e_0-weight μ. Applying a deep result from [CMZ97] one can show that it is enough to check $\partial\beta_p(\gamma)(l_\mu) = 0$ for $\mu = 5, 7, \ldots$; it is then necessarily true for l_0. In fact this equation is automatically satisfied for $\mu > p$, while if $\mu \leq p$ it is a polynomial condition on γ. This explains the form of Theorem 1.

The proof of Theorem 2 is similar. By the proof of Theorem 1, we may assume that $\pi_{21} = \beta_p(\gamma)$ and $\pi_{32} = \beta_q(\gamma+p)$. It is not hard to show that the only $\mathfrak{a}$-relative candidates for π_{31} are multiples of $\beta_{p+q}(\gamma)$, and so we come down to checking when $\beta_q(\gamma+p) \cup \beta_p(\gamma)$ is a multiple of $\beta_{p+q}(\gamma)$. Again using [CMZ97], it is enough to check this on l_μ for $0 < \mu \leq p+q$. As before each such μ gives a polynomial condition on γ, but since we are only solving a proportionality here (rather than an equality as in Theorem 1), the cutoffs $p+q = 7, 9$ play the role played by $p = 5, 7$ in Theorem 1. This explains the form of Theorem 2.

The idea of the proof of Theorem 1 in the singular case is to replace $\mathfrak{a}$-relativity by relativity under the Borel subalgebra $\mathrm{Span}\{e_0, e_1\}$ of $\mathfrak{a}$, but we will not go into further details here.

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