Extensions of the Mass 0 Helicity 0 Representation of the Poincare Group

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ABSTRACT.

Wigner's "little group" description of the irreducible representations of the Poincare group associated to the foward light cone is extended to smooth representations of finite length. As an application, we prove that there is a unique indecomposable representation of this group composed of n copies of the mass 0 helicity 0 representation.

Introduction

The purpose of this paper is to construct an equivalence between the category of smooth representations of finite length of a real semidirect product Lie group $G = H \times_s A$ associated to an orbit of H in the dual A^* of A, and a certain category of representations of the semidirect product of the H-stabilizer of the orbit with A. Here A is a vector space and the orbits of H in A^* are locally closed. A proof that the functor we construct is actually a category equivalence will appear elsewhere (4). As an application of this result, we prove that there is a unique indecomposable representation $\mathcal{V}^n$ of the Poincare group composed of $n+1$ copies of the smooth mass 0 helicity 0 representation $\mathcal{V}^0$, for any n. The existence of $\mathcal{V}^n$ was proven independently by A. Guichardet (9) and G. Rideau (11); here we show that $\mathcal{V}^n$ can be realized in the space of smooth compactly supported functions on the n^{th} order infinitesimal neighborhood of the foward light cone. The uniqueness was conjectured to the author by Rideau, and we remark that it may be possible to obtain an alternate proof of it using cohomological methods, starting from the fact that $\mathrm{Ext}^q(\mathcal{V}^0, \mathcal{V}^0)$ is one dimensional for q equal to 0 or 1.

We give a brief history of the problem. The irreducible unitary representations of G are described by Mackey's generalization of Wigner's result for the Poincare group: they are in bijection with pairs consisting of an orbit $\mathcal{O}$ of H in A^* and an irreducible unitary representation of the H-stabilizer of $\mathcal{O}$. Perhaps the most natural representations of finite length of G to study are bounded representations in a Hilbert space composed of unitary irreducible representations, but so far our methods are suited only to the study of smooth representations of G of finite length, composed of irreducible representations acting in the smooth compactly supported sections of H-vector bundles of finite rank over orbits of H in A^*, so we have begun with these. One finds in the introduction to the paper (7) of Guichardet that if the requirement of compact support is dropped, the same results are obtained, but if one takes the C^∞ vectors of irreducible unitary representations as composition series elements, somewhat pathological results can occur (the example given is due to Blanc (1)).

In the case of the Poincare group, to my knowledge representations such as we study were first considered by Rideau, one of whose results is that there is a unique indecomposable representation composed of two mass 0 helicity 0 representations (10). Later Guichardet generalized Rideau's work to any G as above by constructing an exact sequence arising from representations of length 2 (7), and F. du Cloux made a study of the Ext^n groups in this setting (5). A corollary of Guichardet's work is that a smooth indecomposable representation of finite length is composed of irreducible representations which are all associated to the same orbit (this is false in the example of Blanc's), so here we shall consider only the catgory of representations of finite length composed of irreducible representations associated to a fixed orbit $\mathcal{O}$. In (8) Guichardet obtains a complete description of this category when the tangent bundle of $\mathcal{O}$ has an H-complement in the flat bundle $\mathcal{O} \times A^*$. His result is a generalization of the little group functor, which we generalize here for all orbits.

Rideau's work was built upon in another direction by Cassinelli, Truini, and Varadarajan, who observed that the indecomposable representation composed of two mass 0 helicity 0 representations of the Poincare group is realized in the smooth compactly supported functions on the first order infinitesimal neighborhood of the foward light cone (2). This enabled them to complete it to a bounded representation in a Hilbert space composed of two unitary mass 0 helicity 0 representations. It would be of interest to generalize their completion to all lengths, and also to generalize their observation to see which representations of finite length associated to a fixed orbit can be realized in vector bundles over infinitesimal neighborhoods of the orbit.

We have organized this paper so that in Section 1 we construct the functor, and in Section 2 we apply it to extensions of the mass 0 helicity 0 representation of the Poincare group.

1. Representations of Finite Length of Semidirect Product Lie Groups

We begin by defining the irreducible representations composing our representations of finite length. We use Schwartz' notations $\mathcal{E}$ and $\mathcal{D}$ in place of C^∞ and C_c^∞, and when the argument of $\mathcal{E}$ or $\mathcal{D}$ is a vector bundle it is understood that we are referring to its sections.

DEFINITIONS

Let K a Lie group, $\mathcal{O}$ a homogeneous space for K. We define the category $\mathrm{Geo}_K\mathcal{O}$ of *geometric* representations of K associated to $\mathcal{O}$ as follows. An object is the canonical representation $\mathcal{U}$ of K in $\mathcal{D}(B)$, where B is a complex K-homogeneous vector bundle of finite rank over $\mathcal{O}$: if k in K, p in $\mathcal{O}$, s in $\mathcal{D}(B)$, then $\mathcal{U}_k s(p) = ks(k^{-1}p)$. We say $\mathcal{U}$ is the representation *associated* to B. A morphism between two objects $\mathcal{U}_1$,

$\mathcal{U}_2$ associated to bundles B_1, B_2 is a smooth section of $\mathrm{Hom}(B_1, B_2)$ intertwining their actions.

When S is the K-stabilizer of some point p_0 in $\mathcal{O}$ (the "little group" of $\mathcal{O}$), the restriction functor $\mathcal{R}_S^K$ from $\mathrm{Geo}_K\mathcal{O}$ to the category of complex finite dimensional representations of S is defined as follows. If $\mathcal{U}$ and B are as above, the action of S on B leaves the fiber B_{p_0} invariant, and $\mathcal{R}_S^K\mathcal{U}$ is the resulting representation of S on it. It is elementary that $\mathcal{R}_S^K$ is an equivalence of categories whose inverse is the smooth induction functor Ind_S^K (13).

Let $G = H \times_s A$ be as in the introduction, and let $\mathcal{O}$ be an orbit of H in A^*. We have assumed $\mathcal{O}$ locally closed, so it is a submanifold of A^*. We remark that this is the setting in which the Mackey machine describes the irreducible unitary representations of G (6; 12). Let p_0 in $\mathcal{O}$, S the H-stabilizer of p_0. Notice that $\mathcal{O}$ is a G-space under the trivial action of A, so we have the categories $\mathrm{Geo}_H\mathcal{O}$ and $\mathrm{Geo}_G\mathcal{O}$. We will always view $\mathrm{Geo}_H\mathcal{O}$ as a full subcategory of $\mathrm{Geo}_G\mathcal{O}$ as follows. If B is an H-vector bundle over $\mathcal{O}$ and $\mathcal{U}$ is the associated representation in $\mathrm{Geo}_H\mathcal{O}$, then for a in A, s in $\mathcal{D}(B)$, and p in $\mathcal{O}, \mathcal{U}$ is extended to A by the character action $(\mathcal{U}_a s)(p) = e^{i\langle a,p\rangle}s(p)$. The G-stabilizer of p_0 is SA, and the above injection goes down to the little groups as the map from finite dimensional representations of S to finite dimensional representations of SA given by tensoring with e^{ip_0}.

Let $\mathrm{Ext}_G\mathcal{O}$ be the category of *extensions* of representations in $\mathrm{Geo}_H\mathcal{O}$: objects are C^∞ representations of G, given with a specified finite topologically split composition series of representations in $\mathrm{Geo}_H\mathcal{O}$, and morphisms are continuous linear intertwining maps (which need not respect the specified composition series).

Similarly, for any group S let Ext_S be the category whose objects are complex finite dimensional representations of S given with a specified composition series of representations, and whose morphisms are intertwining maps.

We remark that it is clumsy and unnecessary to to require that representations be given with a composition series; in (4) this requirement is removed. Note that composition series elements are not required to be irreducible; nevertheless, it is clear that objects in $\mathrm{Ext}_G\mathcal{O}$ and Ext_S are of finite length.

EXTENSIONS AS TRIANGULAR MATRICES

Fix n, and for $0 \le i \le n$ let $\mathcal{U}^i$ be representations of $\mathrm{Geo}_H\mathcal{O}$ associated to H-bundles F_i over $\mathcal{O}$, and let $\mathcal{U}$ in $\mathrm{Ext}_G\mathcal{O}$ have (topologically split) composition series $(\mathcal{U}^0, \ldots, \mathcal{U}^n)$. Then up to equivalence $\mathcal{U}$ acts in $\oplus_0^n\mathcal{D}(F_i)$, leaving the flag $\oplus_j^n\mathcal{D}(F_i)$ invariant and defining $\mathcal{U}^j$ in the subquotient $\mathcal{D}(F_j)$. Thus we may write $\mathcal{U}$ as a lower triangular matrix $\mathcal{U}^{ij} : \mathcal{D}(F_j) \to \mathcal{D}(F_i)$, where $\mathcal{U}^{ii} = \mathcal{U}^i$.

Similarly, if $\mathcal{U}'$ is another representation in $\mathrm{Ext}_G\mathcal{O}$ acting in $\oplus_0^{n'}\mathcal{D}(F_i')$ and $T : \mathcal{U} \to \mathcal{U}'$ is a morphism, then T is a n' by n matrix $T^{ij} : \mathcal{D}(F_j) \to \mathcal{D}(F_i')$. In general, $\mathcal{U}^{ij}$ $(i > j)$ and T^{ij} can be complicated continuous linear maps, but our first theorem shows that up to equivalence they are differential operators.

DIFFERENTIAL OPERATORS

Let B, C, and E be vector bundles over $\mathcal{O}$, f and f' diffeomorphisms of $\mathcal{O}$, s in $\mathcal{D}(B)$, p in $\mathcal{O}$. We define *differential operators of order* $\leq m$ *above* f *from* $\mathcal{D}(B)$ *to* $\mathcal{D}(C)$ (we sometimes say from B to C) to be continuous linear maps $F : \mathcal{D}(B) \to \mathcal{D}(C)$ such that $Fs(p)$ depends only on the m-jet of s at $f^{-1}(p)$. This definition behaves well with respect to composition: if F' is an order $\leq m'$ differential operator above f' from C to E, $F' \circ F$ is an order $\leq m' + m$ differential operator above $f' \circ f$ from B to E. Note that if f is the identity, F is an ordinary order $\leq m$ differential operator from B to C, and if $m = 0$, F defines a unique fiberwise linear map $\tilde{F} : B \to C$ mapping $B_p \to C_{f(p)}$ such that $Fs(p) = \tilde{F}\big(s \circ f^{-1}(p)\big)$. When $m = 0$ we refer to F as a *bundle map above* f.

Let $\Delta^r B$ be the vector bundle over $\mathcal{O}$ of order $\leq r$ differential operators on B: a smooth section of $\Delta^r B$ is an order $\leq r$ differential operator above the identity from $\mathcal{D}(B)$ to $\mathcal{D}(\mathcal{O})$. Let f and F be as above, and write $\lambda_f : \mathcal{D}(\mathcal{O}) \to \mathcal{D}(\mathcal{O})$ for the map $\beta \mapsto \beta \circ f^{-1}$, β in $\mathcal{D}(\mathcal{O})$; then λ_f is a bundle map above f. For $r \geq m$, D in $\mathcal{D}(\Delta^{r-m} C)$, the map $\lambda_{f^{-1}} \circ D \circ F : \mathcal{D}(B) \to \mathcal{D}(\mathcal{O})$ is an order $\leq r$ differential operator above the identity, and the map $D \mapsto \lambda_{f^{-1}} \circ D \circ F$ is a bundle map above f^{-1} from $\Delta^{r-m} C$ to $\Delta^r B$.

We remark that when $r = m$ the converse holds: if $\overline{F}$ is a bundle map above f^{-1} from $\Delta^0 C = C^*$ to $\Delta^r B$, there is a unique order $\leq r$ differential operator F above f from B to C such that for all ω in $\mathcal{D}(C^*)$, $\lambda_{f^{-1}} \circ \omega \circ F = \overline{F}\omega$. This fails for $r \geq m$, and one of the main difficulties of this paper arises from the problem of telling when a bundle map above f^{-1} from $\Delta^{r-m} C$ to $\Delta^r B$ arises from an order $\leq m$ differential operator above f as above.

RESULTS

Fix a real subbundle C of the flat bundle $\mathcal{O} \times A^*$, complementary to the tangent bundle $T\mathcal{O}$.

Theorem 1 (3). *Let* $\mathcal{U}$ *and* $\mathcal{U}'$ *be objects and* $T : \mathcal{U} \to \mathcal{U}'$ *a morphism in* $\mathrm{Ext}_G\mathcal{O}$ *as above,* h *in* H*,* a *in* A*,* s *in* $\oplus_0^n \mathcal{D}(F_i)$*. Then for each* i, j*, up to equivalence* $\mathcal{U}_h^{ij}$ *is an order* $\leq i - j$ *differential operator above* $h : \mathcal{O} \to \mathcal{O}$*, and* $\mathcal{U}_a$ *is a bundle map above the identity (note* a *acts as the identity on* $\mathcal{O}$*) of the following special form: for each* a *there is a continuous endomorphism* l_a *of* $\oplus_0^n \mathcal{D}(F_i)$*, linear in* a*, such that* $l_a^{ij} = 0$ *if* $i \leq j$*,* l_a^{ij} *is a bundle map above the identity if* $i > j$*,* $l_a(p) = 0$ *if* $\langle a, C_p \rangle = 0$*, and*

$$\mathcal{U}_a(p) = e^{i\langle a,p\rangle} \exp l_a(p).$$

Note that $\exp l_a$ *is polynomial in* l_a*, as* l_a^{ij} *is strictly lower triangular.*

If $\mathcal{U}$ *and* $\mathcal{U}'$ *are both of this form, the entries* T^{ij} *of the morphism* T *are bundle maps above the identity.*

Since all representations in $\mathrm{Ext}_G\mathcal{O}$ are equivalent to representations as in theorem 1, henceforth we shall fix C and restrict $\mathrm{Ext}_G\mathcal{O}$ to such representations and their morphisms. Roughly, our goal is to extend the domain of the little group functor $\mathcal{R}_S^H$ from $\mathrm{Geo}_H\mathcal{O}$ to $\mathrm{Ext}_G\mathcal{O}$. In general H may act by differential operators, so representations in $\mathrm{Ext}_G\mathcal{O}$ are not induced even from the inhomogeneous little group SA; the main point of this paper is to overcome this problem. However, when C can be chosen to be H-covariant, representations of $\mathrm{Ext}_G\mathcal{O}$ are induced from SA:

Theorem 2 (8). *Let $\mathcal{U}$ in $\mathrm{Ext}_G\mathcal{O}$, with composition series $\mathcal{U}^0,\ldots,\mathcal{U}^n$. If h in H leaves C invariant, then the $\mathcal{U}_h^{ij}$ are bundle maps above h. Consequently, if C is H-covariant and we define $\tilde{S} = S \times_s T\mathcal{O}_{p_0}^{\perp}$, then $\mathrm{Ext}_G\mathcal{O}$ is a full subcategory of $\mathrm{Geo}_G\mathcal{O}$, and is equivalent to the full subcategory of $\mathrm{Ext}_{\tilde{S}}$ of objects having a composition series of representations such that $T\mathcal{O}_{p_0}^{\perp}$ acts by e^{ip_0}. The equivalence is given by $\mathcal{R}_{SA}^G$ followed by ordinary restriction from SA to $\tilde{S}$, applied both to $\mathcal{U}$ and to its specified composition series $\mathcal{U}^0,\ldots,\mathcal{U}^n$.*

When C cannot be chosen to be H-covariant, we proceed as follows. Let $\mathcal{U}$ in $\mathrm{Ext}_G\mathcal{O}$ have composition series $\mathcal{U}^0,\ldots,\mathcal{U}^n$ associated to bundles F_i as above. Then $\mathcal{U}^i$ gives rise to a representation $\Delta^r\mathcal{U}^i$ on $\mathcal{D}(\Delta^r F_i)$: if D in $\mathcal{D}(\Delta^r F_i)$ is an order $\leq r$ differential operator on F_i,

$$(\Delta^r\mathcal{U}^i)_g D = \lambda_g \circ D \circ \mathcal{U}^i_{g^{-1}}$$

(recall that for β in $\mathcal{D}(\mathcal{O})$, $\lambda_g\beta = \beta\circ g^{-1}$). Since $(\Delta^r\mathcal{U}^i)_g$, for which we write $\Delta^r_g\mathcal{U}^i$, is a bundle map above g, the representation $\Delta^r\mathcal{U}^i$ is in $\mathrm{Geo}_G\mathcal{O}$ (although not $\mathrm{Geo}_H\mathcal{O}$). The action of the complexified Lie algebra $\mathfrak{g}$ of G (all our Lie algebras are complexified) under $\mathcal{U}$ gives rise to the following well known description of $\mathcal{R}^G_{SA}\Delta^r\mathcal{U}^i$, the representation of SA on the fiber $\Delta^r F_i(p_0)$. Let $\mathfrak{U}(\mathfrak{g})$ be the universal enveloping algebra of $\mathfrak{g}$ with the usual filtration $\mathfrak{U}_r(\mathfrak{g})$, and let

$$\overline{\mathfrak{U}}_r = \mathfrak{U}_r(\mathfrak{g})\mathfrak{U}(\mathfrak{s}\oplus\mathfrak{a}).$$

For convenience, let $\mathfrak{s}' = \mathfrak{s}\oplus\mathfrak{a}$. Then for Z in $\overline{\mathfrak{U}}_r$, $\mathcal{U}^i_Z$ is a differential operator above the identity from F_i to F_i of order $\leq r$ at p_0, and for ω in $F_i^*(p_0)$, $\omega\otimes Z \mapsto \omega\circ\mathcal{U}^i_Z$ defines a projection

$$F_i^*(p_0)\otimes\overline{\mathfrak{U}}_r \to \Delta^r F_i(p_0)$$

that factors to an isomorphism from $F_i^*(p_0)\otimes_{\mathfrak{s}'}\overline{\mathfrak{U}}_r$ to $\mathcal{R}^G_{SA}(\Delta^r\mathcal{U}^i)$. This projection intertwines both the $(\mathcal{R}^G_{SA}\mathcal{U}^i)^*\otimes\mathrm{Ad}$ action of SA and the right action of $\mathfrak{s}'$ on $F_i^*\otimes\overline{\mathfrak{U}}_r$ with $\mathcal{R}^G_{SA}\Delta^r\mathcal{U}^i$. It is because we will be generalizing the right action of $\mathfrak{s}'$ below that we use $\overline{\mathfrak{U}}_r$ instead of the more usual $\mathfrak{U}_r(\mathfrak{g})$ here.

Recall that for all g in G, $\mathcal{U}^{ij}_g$ is an order $\leq i-j$ differential operator above $g:\mathcal{O}\to\mathcal{O}$. Consider the vector bundle $\oplus_0^n\Delta^{n-i}F_i$. We say that a smooth section D is a *graded differential operator*, as when restricted to $\oplus_j^n\mathcal{D}(F_i)$ it is of order $\leq n-j$. We define a representation $\Delta^{\mathrm{gr}}\mathcal{U}$ of G on $\oplus_0^n\mathcal{D}(\Delta^{n-i}F_i)\ni D$ such that the action of $(\Delta^{\mathrm{gr}}\mathcal{U})_g$, which we write as $\Delta^{\mathrm{gr}}_g\mathcal{U}$, is

$$\Delta^{\mathrm{gr}}_g\mathcal{U}(D) = \lambda_g\circ D\circ\mathcal{U}_{g^{-1}}.$$

It is clear that $\Delta^{\mathrm{gr}}\mathcal{U}$ is a representation; one checks that it acts by bundle maps above g and has composition series $\{\Delta^0\mathcal{U}^n,\ldots,\Delta^n\mathcal{U}^0\}$ (note the order reversal), so it lies in both $\mathrm{Geo}_G\mathcal{O}$ and $\mathrm{Ext}_G\mathcal{O}$ (for $i\neq 0$, the composition series element $\Delta^i\mathcal{U}^{n-i}$

is not in $\mathrm{Geo}_H\mathcal{O}$, but it has a canonical composition series of representations on symbol bundles that are). If $T:\mathcal{U}\to\mathcal{U}'$ is a morphism, T is a bundle map, and for D' in the space where $\Delta^{\mathrm{gr}}\mathcal{U}'$ acts, we define $\Delta^{\mathrm{gr}}T(D') = D'\circ T$; this makes Δ^{gr} a contravariant functor, and we have the following:

Theorem 3 (4). *The functor $\mathcal{R}^G_{SA}\circ\Delta^{\mathrm{gr}}$ from $\mathrm{Ext}_G\mathcal{O}$ to Ext_{SA}, mapping the object $\mathcal{U}$ with composition series $\mathcal{U}^0,\ldots,\mathcal{U}^n$ to the object $\mathcal{R}^G_{SA}\circ\Delta^{\mathrm{gr}}\mathcal{U}$ with composition series $\mathcal{R}^G_{SA}\Delta^0\mathcal{U}^n,\ldots,\mathcal{R}^G_{SA}\Delta^n\mathcal{U}^0$, is contravariant and faithful.*

Recall that a functor is faithful if it is injective on morphisms. We will spend the rest of Section 1 describing the image of $\mathcal{R}^G_{SA}\circ\Delta^{\mathrm{gr}}$, which we do by generalizing the description of $\mathcal{R}^G_{SA}\Delta^r\mathcal{U}^i$ above.

Lemma 4 (4). *Let $\mathcal{U}$ in $\mathrm{Ext}_G\mathcal{O}$, X in $\mathfrak{h}$. The lower triangular matrix $\mathcal{U}^{ij}_X$ is an order ≤ 1 differential operator above the identity.*

We remark that the proof depends on the special form of the $\mathcal{U}$-action of A given in theorem 1. Using only the graded form of the action of H leads only to $\mathcal{U}^{ij}_X$ order $\leq i-j$ for $i>j$, order ≤ 1 for $i=j$.

If X in $\mathfrak{s}$, it is clear that $\mathcal{U}^{ii}_X = \mathcal{U}^i_X$ is order 0 at p_0, but for $i>j$, $\mathcal{U}^{ij}_X$ may be order 1 at p_0. Indeed, $\mathcal{U}$ itself is induced from SA if and only if $\mathcal{U}^{ij}_X$ is order 0 at p_0 for all X in $\mathfrak{s}$, $i\geq j$. Observe that if α is in $\mathfrak{a}=A\otimes\mathbf{C}$, $\mathcal{U}_\alpha$ is order 0 by theorem 1: $\mathcal{U}_\alpha(p) = i\langle\alpha,p\rangle + l_\alpha(p)$.

In order to describe the image of $\mathcal{R}^G_{SA}\circ\Delta^{\mathrm{gr}}$, we must define several flags of vector spaces built from the fibers $F_i(p_0)$. For $0\leq j\leq n$, let

$$E_j = \bigoplus_0^j \Delta^{j-i}F_i(p_0), \qquad C_j = \bigoplus_0^j F_i^*(p_0)\otimes\overline{\mathfrak{U}}_{j-i}.$$

The flag $C_0\subset\ldots\subset C_n$ is a split $\mathfrak{s}'$ representation under the right action of $\mathfrak{s}'$ on $\overline{\mathfrak{U}}_{j-i}$. Let us write σ for the representation $\mathcal{R}^G_{SA}\circ\Delta^{\mathrm{gr}}\mathcal{U}$; then σ acts in E_n so as to leave E_j invariant, and $\mathcal{U}$ defines an $\mathfrak{s}'$-projection of flags $C_j\to E_j$:

$$\omega\otimes Z\mapsto\omega\circ\mathcal{U}_Z, \quad \omega\in F_i^*(p_0), Z\in\overline{\mathfrak{U}}_{j-i}, 0\leq i\leq j\leq n.$$

Note that this projection respects the direct sum structure of the C_j and the E_j if and only if $\mathcal{U}$ is the split representation $\oplus_0^n\mathcal{U}^i$, in which case it is the sum of the projections describing $\mathcal{R}^G_{SA}\Delta^{n-i}\mathcal{U}^i$ above.

Let K_j be the kernel of the projection $C_j\to E_j$; then K_j is a $\mathfrak{s}'$-subspace of C_j, and the problem of describing σ amounts to that of describing the flag $\{K_j\}$. In order to do so it is convenient to fix a cross section for E_j in C_j.

Choose a subspace $\mathfrak{m}$ of $\mathfrak{h}$ complementary to $\mathfrak{s}$, let $(m_1,\ldots,m_p)$ be an ordered basis of $\mathfrak{m}$, and let M_r be the finite dimensional subspace of $\overline{\mathfrak{U}}_r$ spanned by the ordered monomials in the m_k of degree $\leq r$. Then it is well known that $F_i^*(p_0)\otimes M_r$ is a cross section for $\Delta^rF_i(p_0)$ in $F_i^*(p_0)\otimes\overline{\mathfrak{U}}_r$, and similarly one proves by induction that if we define

$$D_j = \bigoplus_0^j F_i^*(p_0)\otimes M_{j-i},$$

then $C_j = D_j \oplus K_j$, independent of $\mathcal{U}$.

At this point, notice that the endomorphism $\Delta_g^{\mathrm{gr}}\mathcal{U}$ of $\oplus_0^n \mathcal{D}(\Delta^{n-i}F_i)$ is completely determined by its restriction to the (non-invariant) subspace $\oplus_0^n \mathcal{D}(F_i^*)$, because it comes from the differential operator $\mathcal{U}_{g^{-1}}$. Less obviously, an analogous result holds at the fiber p_0: for $0 \le j \le n$ let

$$B_j = \bigoplus_0^j F_i^*(p_0),$$

and note that B_j is in both E_j and C_j, and that the projection $C_j \to E_j$ is the identity on it. The representation σ is determined by its restriction to the (non-invariant) subspace B_j, as follows.

For each X in $\mathfrak{s}'$, ω in B_n, we define a linear map

$$\eta_X : B_n \to D_n$$

by defining $\eta_X \omega$ to be the unique element of D_n that projects to $\sigma_X \omega = -\omega \circ \mathcal{U}_X$ in E_n. We define also a generalization of the right action of $\mathfrak{U}(\mathfrak{s}') = \overline{\mathfrak{U}}_0$ on C_j: an element Z' of $\overline{\mathfrak{U}}_{k-j}$ defines a map from $C_j \ni \omega \otimes Z$ to C_k by

$$(\omega \otimes Z)Z' = \omega \otimes (ZZ').$$

Lemma 5 (4). *For all X in $\mathfrak{s}'$, $\eta_X(B_j)$ is in D_j, so η is a linear map from $\mathfrak{s}'$ to* $\mathrm{Hom}\big(\{B_j\}, \{D_j\}\big)$. *Furthermore η, which is the lift to D_j of the restriction of σ to B_j, determines K_j and hence σ*:

$$K_j = \mathrm{Span}_{\mathbf{C}}\big\{\omega \otimes XZ + (\eta_X \omega)Z : \omega \in B_i,\ X \in \mathfrak{s}',\ Z \in \overline{\mathfrak{U}}_{j-i},\ 0 \le i \le j \le n\big\}.$$

Now any linear map $\tilde{\eta} : \mathfrak{s}' \to \mathrm{Hom}\big(\{B_j\}, \{D_j\}\big)$ defines $\mathfrak{s}'$-subspaces $\tilde{K}_j$ of C_j as above, and thus a representation $\tilde{\sigma}$ of $\mathfrak{s}'$ on the flag $\{C_j/\tilde{K}_j\}$. It turns out that a representation such as $\tilde{\sigma}$ is in the image of $\mathcal{R}_{SA}^G \circ \Delta^{\mathrm{gr}}$ essentially if and only if $D_j \oplus \tilde{K}_j = C_j$ and $\tilde{\sigma}$ lifts from $\mathfrak{s}'$ to SA. We make this precise by defining the subcategory Gr_{SA} of Ext_{SA} that is the image of $\mathrm{Ext}_G\mathcal{O}$ under the functor $\mathcal{R}_{SA}^G \circ \Delta^{\mathrm{gr}}$.

An object of Gr_{SA} is a complex finite dimensional representation σ of SA, given with specified vector spaces $W_0, \ldots, W_n$ and a certain map η which defines σ as follows. For $0 \le j \le n$, define flags C_j, D_j, and B_j as above, but with the W_i replacing the $F_i^*(p_0)$. The map η is linear from $\mathfrak{s}'$ to $\mathrm{Hom}\big(\{B_j\}, \{D_j\}\big)$, and it defines $\mathfrak{s}'$-subspaces K_j of the C_j just as above. We require η to be such that $K_j \oplus D_j = C_j$ for all j, and the representation of $\mathfrak{s}'$ in C_n/K_n lifts to SA; this lift is σ. We require also that for any α in $\mathfrak{a}$, $\eta_\alpha(B_j)$ is in B_j, the action of η_α on the subquotients $B_j/B_{j-1} = W_j$ is multiplication by $-i\langle \alpha, p_0\rangle$, and if $\langle \alpha, C_{p_0}\rangle = 0$, then $\eta_\alpha = -i\langle \alpha, p_0\rangle$ (compare to the action of $\mathcal{U}|_A$ in theorem 1).

If $\{\sigma', W_0', \ldots, W_{n'}', \eta'\}$ is another object of Gr_{SA}, a morphism between the two is a linear map $\tau : B_n \to B_{n'}'$ such that for all X in $\mathfrak{s}'$,

$$(\tau \otimes 1) \circ \eta_X = \eta_X' \circ \tau,$$

where $(\tau \otimes 1)(\omega \otimes Z) = \tau\omega \otimes Z$. Then $\tau \otimes 1 : C_n \to C'_{n'}$ maps K_n into $K'_{n'}$ and projects to an intertwining map $t : \sigma \to \sigma'$ (compare to the fact that morphisms between objects of $\mathrm{Ext}_G\mathcal{O}$ as in theorem 1 are order 0 differential operators).

The contravariant functor $\mathcal{R}^G_{SA} \circ \Delta^{\mathrm{gr}}$ from $\mathrm{Ext}_G\mathcal{O}$ to Gr_{SA} takes an object $\{\mathcal{U}, \mathcal{U}^0, \ldots, \mathcal{U}^n\}$ to the object $\{\sigma, F_0^*(p_0), \ldots, F_n^*(p_0), \eta\}$, where η and σ are defined from $\mathcal{U}$ as above, and a morphism $T : \mathcal{U} \to \mathcal{U}'$ to the dual $T^*(p_0) : B'_{n'} \to B_n$ of $T(p_0)$.

Theorem 6 (4). *The functor $\mathcal{R}^G_{SA} \circ \Delta^{\mathrm{gr}}$ from $\mathrm{Ext}_G\mathcal{O}$ to Gr_{SA} is a contravariant equivalence of categories.*

In order to apply this theorem, we need two more lemmas. The first gives a useful condition on η guaranteeing that $K_j \oplus D_j = C_j$, and the second restricts the form of η.

Let $W_0, \ldots, W_n$ be arbitrary vector spaces, and define from them C_j, D_j, and B_j as above. Suppose $\tilde{\eta} : \mathfrak{s}' \to \mathrm{Hom}\big(\{B_j\}, \{D_j\}\big)$ is any linear map; we can still use it to define $\mathfrak{s}'$-subspaces $\tilde{K}_j$ of C_j as above. For $\omega \otimes Z$ in C_j, X, Y in $\mathfrak{s}'$, we define endomorphisms $\tilde{\eta}_X \otimes 1$ and $1 \otimes \mathrm{ad}_X$ of C_j by

$$(\tilde{\eta}_X \otimes 1)(\omega \otimes Z) = (\tilde{\eta}_X \omega)Z, \quad (1 \otimes \mathrm{ad}_X)(\omega \otimes Z) = \omega \otimes (XZ - ZX).$$

We define $\theta_{\tilde{\eta}}(X, Y) : B_j \to D_j$ by

$$\theta_{\tilde{\eta}}(X, Y) = (\tilde{\eta}_X \otimes 1 + 1 \otimes \mathrm{ad}_X) \circ \tilde{\eta}_Y - (\tilde{\eta}_Y \otimes 1 + 1 \otimes \mathrm{ad}_Y) \circ \tilde{\eta}_X - \tilde{\eta}_{[X,Y]}.$$

Lemma 7. *For any $\tilde{\eta}$ as above, $\tilde{K}_j + D_j = C_j$ for $0 \le j \le n$. We have $\tilde{K}_j \oplus D_j = C_j$ if and only if for all X, Y in $\mathfrak{s}'$, $0 \le j \le n$, $\theta_{\tilde{\eta}}(X, Y)(B_j)$ is in $\tilde{K}_{j-1}$.*

Lemma 8. *Let $\{\sigma, W_0, \ldots, W_n, \eta\}$ be an object of Gr_{SA}. It is a consequence of the form of $\eta|_{\mathfrak{a}}$ that for all X in $\mathfrak{s}'$ $0 \le j \le n$,*

$$\eta_X(B_j) \subset (B_{j-1} \otimes M_1) \oplus W_j$$

(compare to lemma 4), and if X leaves C_{p_0} invariant, $\eta_X(B_j)$ is in B_j (compare to theorem 2). Furthermore, for each j, η_X projects to a map

$$\eta^j_X : B_j/B_{j-1} \to (B_{j-1} \otimes M_1 \oplus W_j)/(B_{j-1} \otimes M_1).$$

Both of these spaces are canonically isomorphic to W_j, and η^j is a representation of $\mathfrak{s}'$ on W_j. If an object $\mathcal{U}$ with composition series $\mathcal{U}^0, \ldots, \mathcal{U}^n$ in $\mathrm{Ext}_G\mathcal{O}$ goes to $\{\sigma, W_0, \ldots, W_n, \eta\}$ under $\mathcal{R}^G_{SA} \circ \Delta^{\mathrm{gr}}$, then η^j is the dual of the representation $\mathcal{R}^G_{SA}\mathcal{U}^j$.

2. Extensions of the Mass 0 Helicity 0 Representation of the Poincare Group

Throughout this section, let $G = H \times_s A$ be the universal cover of the Poincare group: $H = \mathrm{SL}_2\mathbf{C}$ acts on $A = \mathbf{R}^{1,3}$ by the standard double cover $\mathrm{SL}_2\mathbf{C} \to \mathrm{SO}_{1,3}$. Here the standard inner product of signature $1, 3$ gives an H-isomorphism of A^*

with A, so we view A^* as equal to A. The theorem of Guichardet's (theorem 2 above) describes $\mathrm{Ext}_G\mathcal{O}$ for any H-orbit $\mathcal{O}$ in A except the two light cones. Henceforth let $\mathcal{O}$ be the foward light cone and take p_0 to be $(1,0,0,1)$; then the H-stabilizer S of p_0 is

$$S = \left\{ \begin{pmatrix} z & a \\ 0 & z^{-1} \end{pmatrix} : z \in S^1, a \in \mathbf{C} \right\}.$$

Let $\mathcal{V}$ be the representation in $\mathrm{Geo}_H\mathcal{O}$ induced from the trivial representation of S, so that $\mathcal{V}$ is the canonical representation of G in $\mathcal{D}(\mathcal{O})$.

Theorem 9 (4). *Up to equivalence there is a unique indecomposable representation in* $\mathrm{Ext}_G\mathcal{O}$ *composed of* $n+1$ *copies of* $\mathcal{V}$*, for all* $n \geq 0$.

Remark. As stated in the introduction, the existence was proven independently by Guichardet (9) and Rideau (11), and it may be possible to prove the uniqueness by cohomological methods. Our approach is to apply the "little group method" for describing representations of finite length associated to the foward light cone that was developed in Section 1.

Outline of Proof. Suppose $\mathcal{U}$ together with its composition series $\mathcal{U}^0, \ldots, \mathcal{U}^n$ is an object of $\mathrm{Ext}_G\mathcal{O}$, and $\mathcal{U}^i = \mathcal{V}$ for all i. The representation $\mathcal{V}$ is associated to the trivial bundle $\mathcal{O} \times \mathbf{C}$ over $\mathcal{O}$, and so the functor $\mathcal{R}^G_{SA} \circ \Delta^{\mathrm{gr}}$ maps $\mathcal{U}$ with its composition series to an object $\{\sigma, W_0, \ldots, W_n, \eta\}$ of Gr_{SA}, where W_i is $\mathbf{C}$ for all i. Recall that if we define C_j, D_j, and B_j as before, then η defines the $\mathfrak{s}' = \mathfrak{s} \oplus \mathfrak{a}$-subspaces K_j of C_j, and $\sigma = \mathcal{R}^G_{SA} \circ \Delta^{\mathrm{gr}}\mathcal{U}$ is the representation of $\mathfrak{s}'$ in C_n/K_n, lifted to SA.

Here $\mathcal{R}^G_{SA}\mathcal{U}^i$ restricts to the trivial representation of S for all i, and so by lemma 8 the representations $\eta^j|_{\mathfrak{s}}$ on $W_j = \mathbf{C}$ are all trivial, or in other words for $0 \leq j \leq n$, X in $\mathfrak{s}$,

$$\eta_X(B_j) \subset B_{j-1} \otimes M_1.$$

Since $\mathcal{R}^G_{SA} \circ \Delta^{\mathrm{gr}}$ is an equivalence of categories, we need only prove that there is a unique such indecomposable object of Gr_{SA}.

Let e_0, e_1, e_2, e_3 be the standard orthogonal basis of $\mathbf{R}^{1,3}$: $\langle e_0, e_0 \rangle = 1$, $\langle e_i, e_i \rangle = -1$ for $i > 0$. For a basis of $\mathfrak{a} = A \otimes \mathbf{C}$ we take the vectors

$$x_0 = p_0 = e_0 + e_3, x_3 = e_0 - e_3, x_+ = e_1 - ie_2, x_- = e_1 + ie_2.$$

In Section 1 we had to choose a real vector bundle C complementary to $T\mathcal{O}$ in $\mathcal{O} \times A^*$, but here we need only choose C_{p_0}. The vectors x_0, e_1, e_2 are a basis of $T\mathcal{O}_{p_0}$, so we may take C_{p_0} to have basis x_3. Then the annihilator $C^{\perp}_{p_0}$ of C_{p_0} in $\mathfrak{a}$ has basis x_+, x_-, x_3, and by the definition of Gr_{SA}, $\eta_a = -i\langle a, x_0 \rangle$ for a in $C^{\perp}_{p_0}$, and

$$\eta_{x_0} = -i\langle x_0, x_0 \rangle + N = N,$$

where N is a (nilpotent) endomorphism of B_n such that $N(B_j)$ is in B_{j-1}.

We must now describe $\mathfrak{s} \subset \mathfrak{h}$, which we identify with $\mathfrak{so}_{1,3}$. Let $e^*_0, \ldots, e^*_3$ be the basis of A^* dual to $e_0, \ldots, e_3$ (of course, under the canonical isomorphism $A = A^*$, $e_0 = e^*_0$, etc.). The usual basis of $\mathfrak{h}$ is

$$M_{0j} = e_0 \otimes e^*_j + e_j \otimes e^*_0, \quad L_{ij} = e_i \otimes e^*_j - e_j \otimes e^*_i,$$

where $1 \leq i, j \leq 3$. Another basis of $\mathfrak{h}$ is given by the vectors

$$L = 2iL_{12}, L_{\pm} = L_{13} \mp iL_{23}, M_{\pm} = M_{01} \mp iM_{02}, M_3 = M_{03}.$$

The vectors

$$L, X = (M_+ - L_+)/2, Y = (M_- - L_-)/2$$

are a basis of $\mathfrak{s}$, and we will take M_+, M_-, and M_3 as our ordered basis of $\mathfrak{m}$ (in fact, in light of lemma 8 we really need only specify the space $\mathfrak{m}$).

We must calculate the possibilities for $\eta|_{\mathfrak{s}}$. Since C_{p_0} is L-invariant, lemma 8 gives that $\eta_L(B_n)$ is in B_n, and so $\eta_L = \sigma_L|_{B_n}$. Now the representations η^j of lemma 8 are trivial here, and σ lifts to SA, so σ_L is semisimple and hence $\eta_L = 0$.

We next consider η_X. Notice that $M_{\pm}$ and M_3 are eigenvectors of ad_L with eigenvalues ± 2, 0, respectively. From this one checks that if θ_η is as in lemma 7, then $\theta_\eta(L, X)$ maps B_n to D_n. But by lemma 7 its image is in K_n, so it is 0. Since $[L, X] = 2X$,

$$0 = \theta_\eta(L, X) = (1 \otimes \mathrm{ad}_L) \circ \eta_X - 2\eta_X,$$

i.e. η_X maps B_n to a $+2$-eigenspace of $1 \otimes \mathrm{ad}_L$ in D_n. Therefore

$$\eta_X(B_j) \subset B_{j-1} \otimes M_+,$$

and so there is a nilpotent endomorphism γ of B_n sending B_j to B_{j-1} such that for ω in B_n, $\eta_X\omega = \gamma\omega \otimes M_+$. We write $\eta_X = \gamma \otimes M_+$.

We can relate γ and $\eta_{x_0} = N$ by looking at $\theta_\eta(X, x_-)$. Since $[X, x_-] = Xx_- = x_0$, and $\eta_{x_-} = -i\langle x_-, x_0\rangle = 0$, we have for ω in B_j that

$$\theta_\eta(X, x_-)\omega = -(1 \otimes \mathrm{ad}_{x_-}) \circ (\gamma \otimes M_+)\omega - N\omega \in K_{j-1}.$$

Now $\mathrm{ad}_{x_-} M_+ = -M_+x_- = -x_0 - x_3$, so we find that

$$\gamma\omega \otimes (x_0 + x_3) - N\omega \in K_{j-1}.$$

But by definition, $(\eta_{x_0} + \eta_{x_3})\gamma\omega + \gamma\omega \otimes (x_0 + x_3)$ is in K_{j-1}, as ω is in B_{j-1}, so

$$(\eta_{x_0} + \eta_{x_3})\gamma\omega + N\omega = \big((N - 2i)\gamma + N\big)\omega \in K_{j-1}.$$

This also lies in $B_{j-1} \subset D_{j-1}$, so it must be 0. Using N nilpotent gives $\gamma = N(2i - N)^{-1}$.

Similar calculations show that $\{\sigma, W_0, \ldots, W_n, \eta\}$ is an object of Gr_{SA}, such that $W_i = \mathbf{C}$ for all i and the representations $\eta^i|\mathfrak{s}$ of lemma 8 are all trivial, if and only if $\eta_a = -i\langle a, x_0\rangle$ for a in the span of $\{x_\pm, x_3\}$, $\eta_{x_0} = N$ is an arbitrary nilpotent endomorphism of B_n mapping B_j to B_{j-1}, $\eta_L = 0$, $\eta_X = \gamma \otimes M_+$, and $\eta_Y = \gamma \otimes M_-$, where $\gamma = N(2i - N)^{-1}$.

It is easy to see from the definition of Gr_{SA} that such an object decomposes if and only if the action of N on B_n decomposes, and that two such objects η and η' are equivalent if and only if η_{x_0} and η'_{x_0} are similar. This proves theorem 9.

Corollary 10. *The unique indecomposable representation in* $\mathrm{Ext}_G\mathcal{O}$ *composed of* $n+1$ *copies of the mass 0 helicity 0 irreducible representation* $\mathcal{V}$ *is realized in the smooth compactly supported functions on the* n^{th} *order infinitesimal neighborhood of the foward light cone* $\mathcal{O}$.

Outline of Proof. In order to view the canonical representation of G in the space of such functions as an object in $\mathrm{Ext}_G\mathcal{O}$, it is necessary first to define such functions (this is a bit tricky because $\mathcal{O}$ is not closed), and second to use the bundle C of theorem 1 to fix specific transversal derivatives on $\mathcal{O}$; we shall do these things in a later paper. Once they are done, we need only check that $\mathcal{R}^G_{SA} \circ \Delta^{\mathrm{gr}}$ sends the resulting object of $\mathrm{Ext}_G\mathcal{O}$ to an object η of Gr_{SA} such that η_{x_0} is indecomposable, and this is easy.

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