INVARIANT THEORY FOR COINCIDENTAL COMPLEX REFLECTION GROUPS

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Abstract. V.F. Molchanov considered the Hilbert series for the space of invariant skew-symmetric tensors and dual tensors with polynomial coefficients under the action of a real reflection group, and he speculated that it had a certain product formula involving the exponents of the group. We show that Molchanov's speculation is false in general but holds for all *coincidental* complex reflection groups when appropriately modified using exponents and co-exponents. These are the irreducible well-generated (i.e., duality) reflection groups with exponents forming an arithmetic progression and include many real reflection groups and all non-real Shephard groups, e.g., the Shephard-Todd infinite family $G(d, 1, n)$. We highlight consequences for the q-Narayana and q-Kirkman polynomials, giving simple product formulas for both, and give a q-analogue of the identity transforming the h-vector to the f-vector for the coincidental finite type cluster/Cambrian complexes of Fomin–Zelevinsky and Reading. We include the determination of the Hilbert series for the non-coincidental irreducible complex reflection groups as well.

1. Introduction

Molchanov [24] hypothesized a formula for the dimensions of invariants of certain finite real reflection groups acting on skew-symmetric tensors and dual tensors with polynomial coefficients. His formula gives evidence that Solomon's invariant theory [35] for differential forms may have an extension to mixed derivation differential forms. We examine these forms not just for real reflection groups, but for complex reflection groups in general and compute their Hilbert series. We reformulate Molchanov's hypothesis in terms of exponents and coexponents of the group. Although his formula and this reformulation do not hold for all real reflection groups, they do hold for the important class of *coincidental reflection groups*.

The invariant theory of reflection groups acting on $V = \mathbb{C}^n$ displays a wondrous numerology controlled by two sequences of positive integers, the *exponents* $e_1 \le e_2 \le \cdots \le e_n$ and *coexponents* $e_1^* \le e_2^* \le \cdots \le e_n^*$. Solomon's Theorem [35] gives the dimensions of W-invariant polynomial differential forms on V entirely in terms of the exponents of the reflection group W; his proof extends to describe likewise the invariant derivation forms in terms of the coexponents (see [25]). The *degrees* are the integers $d_i = e_i + 1$. Those reflection groups satisfying $e_i + e^*_{n+1-i} = d_n$ are called *duality groups*. These are precisely the *well-generated* reflection groups, i.e., those generated by $n = \dim V$ reflections, and include all Coxeter groups.

An irreducible duality group W is *coincidental* if its exponents $(e_1, e_2, \ldots, e_n)$ form an arithmetic sequence $(e_1, e_1 + a, e_1 + 2a, \ldots, e_1 + (n-1)a)$ for some positive integer a which we call its *exponent gap*. The coincidental reflection groups are the Coxeter groups of types A_n, B_n/C_n, $I_2(m)$, H_3, the monomial groups $G(d, 1, n)$, all irreducible duality groups in rank 2, and the groups G_{25}, G_{26}, and G_{32} in the Shephard-Todd classification [31]. They include all non-Coxeter *Shephard groups*, that is, the symmetry groups of regular complex polytopes.

We extend Solomon's description for the Hilbert series for invariant differential forms, $(S(V^*) \otimes \wedge V^*)^W$, to invariant *mixed derivation differential forms*, $(S(V^*) \otimes \wedge V^* \otimes \wedge V)^W$. We use the *elementary symmetric functions* $\sigma_r(x_1, \ldots, x_n) := \sum_{1 \le i_1 < \cdots < i_r \le n} x_{i_1} x_{i_2} \cdots x_{i_r}$ with the convention that $\sigma_0(x_1, \ldots, x_n) \equiv 1$.

Theorem 1.1. *For any coincidental complex reflection group W acting on $V = \mathbb{C}^n$,*

$$\mathrm{Hilb}\left((S(V^*) \otimes \wedge V^* \otimes \wedge V)^W, q, t, s\right) \;=\; \sum_{r=0}^{n} s^r \;\sigma_r(q^{e_1^*}, \ldots, q^{e_n^*})\; \frac{\prod_{i=1}^{r}(1 + q^{-e_i^*} t)\;\prod_{i=1}^{n-r}(1 + q^{e_i} t)}{\prod_{i=1}^{n}(1 - q^{d_i})}$$

where the coefficient of $q^i t^k s^r$ in the Hilbert series is the dimension of $(S^i(V^*) \otimes \wedge^k V^* \otimes \wedge^r V)^W$.

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Here we use the standard grading on $S(V^*) = \oplus_i S^i(V^*)$ by polynomial degree. We may reformulate Theorem 1.1 compactly using the *q-Pochhammer* notation defined by

$$(z;q)_k := (1-z)(1-zq)\cdots(1-zq^{k-1})$$

and the *q-binomial coefficient* defined by

$$\begin{bmatrix} n \\ r \end{bmatrix}_q := \frac{(q;q)_n}{(q;q)_r\,(q;q)_{n-r}}\,. \tag{1.2}$$

For a coincidental reflection group W, since the coexponents $(e_1^*, e_2^* \ldots, e_n^*) = (1, 1+a, 1+2a, \ldots, 1+(n-1)a)$, we can use the well-known identity [21, Chap. I §2, Ex. 3]

$$\sigma_r(1, q, q^2, \ldots, q^{n-1}) = q^{\binom{r}{2}} \begin{bmatrix} n \\ r \end{bmatrix}_q \tag{1.3}$$

to rewrite the r^{th} elementary symmetric function appearing in the theorem as

$$\sigma_r(q^{e_1^*}, \ldots, q^{e_n^*}) = \sigma_r(q^1, q^{1+a}, q^{1+2a}, \ldots, q^{1+(n-1)a}) = q^r \cdot \sigma_r(q^a, q^{2a}, \ldots, q^{(n-1)a}) = q^{r+a\binom{r}{2}} \begin{bmatrix} n \\ r \end{bmatrix}_{q^a}\,.$$

We focus on the summand $\wedge^r V$ in $\wedge V = \oplus_{r=0}^n \wedge^r V$ and give the following equivalent version of Theorem 1.1.

Theorem 1.1$'$. *For a coincidental complex reflection group W with smallest exponent e_1, exponent gap a,*

$$\mathrm{Hilb}\left((S(V^*) \otimes \wedge V^* \otimes \wedge^r V)^W, q, t\right) = q^{r+a\binom{r}{2}} \begin{bmatrix} n \\ r \end{bmatrix}_{q^a} \frac{(-tq^{e_1}; q^a)_{n-r}(-tq^{-1}; q^{-a})_r}{(q^{e_1+1}; q^a)_n} \quad \textit{for } r = 0, \ldots, n.$$

In fact, we compile the data on the Hilbert series of $(S(V^*) \otimes \wedge V^* \otimes \wedge V)^W$ for *all* irreducible complex reflection groups W, not just the coincidental groups—see Section 3 for Shephard and Todd's infinite family $G(de, e, n)$ of monomial groups and Section 11 for the exceptional groups.

The q-analogues of f-vectors and h-vectors. Theorem 1.1$'$ suggests q-analogues of the *f-vector* and the *h-vector* appearing in the algebraic combinatorics of certain simple polytopes and simplicial spheres, as we will explain in Section 10. These vectors record the number of faces of each dimension and the Hilbert-Poincaré polynomial of the associated Stanley-Reisner ring. For a coincidental reflection group W, we rename the right side of Theorem 1.1$'$ as follows:

$$f_r(W; q, t) \;:=\; q^{r+a\binom{r}{2}} \begin{bmatrix} n \\ r \end{bmatrix}_{q^a} \frac{(-tq^{e_1}; q^a)_{n-r}(-tq^{-1}; q^{-a})_r}{(q^{e_1+1}; q^a)_n}\,.$$

We wish to relate $f_r(W; q, t)$ to a second product:

$$h_r(W; q, t) := (-tq^{-ar-1})^{n-r} \begin{bmatrix} n \\ r \end{bmatrix}_{q^a} \frac{(-tq^{-1}; q^{-a})_r}{(q^{e_1+1}; q^a)_r}\,.$$

We will see in Section 10 that the specializations

$$f_r := \left[f_r(W; q, q^{h+1})\right]_{q=1} \quad \text{for Coxeter number} \quad h := e_n + 1$$

give the number of faces of each dimension in the *finite type cluster fans* of Fomin and Zelevinsky [8] when W is a Weyl group, or the *Cambrian fans* of Reading [27] when W is a real reflection group. For simplicial fans or polytopes, a standard re-encoding gives the f-vector entries f_r in terms of the h-vector entries h_r:

$$\sum_{r=0}^{n} s^r f_r \;=\; \sum_{r=0}^{n} (1+s)^r \cdot h_r\,. \tag{1.4}$$

In Section 9, we use Theorem 1.1$'$ to prove a q-analogue (and even a (q, t)-analogue) of this standard encoding:

Theorem 1.5. *For any coincidental reflection group W with exponent gap a,*

$$\sum_{r=0}^{n} s^r f_r(W; q, t) \;=\; \sum_{r=0}^{n} (-sq; q^a)_r \cdot h_r(W; q, t).$$

In Section 10, we explain why specializing t in $f_r(W;q,t)$ and $h_r(W;q,t)$ to certain powers of q give the *q-Catalan numbers*, *q-Kirkman numbers*, and *q-Narayana numbers* arising previously in [2, 29, 36] and how these specialize further to the aforementioned f-vector and h-vector entries.

Outline. After recalling the numerology of reflection groups in Section 2, we show Theorem 1.1 directly for the infinite family $G(d,1,n)$ and the Weyl groups of type A in Section 3 using results of Kirillov and Pak [19] and Koike [20]. We also give the Hilbert series explicitly for the groups $G(de,e,n)$ in Section 3. We conjecture an explicit basis for $(S(V^*)\otimes \wedge V^* \otimes \wedge V)^W$ in Section 4 constructed from invariant differential operators for coincidental reflection groups; invariance of the alleged basis elements in Conjecture 4.1 is checked in Section 5. In Section 6, we use the Gutkin-Opdam Lemma to predict the sum of degrees of these alleged basis elements. Section 7 then outlines the proof of Theorem 1.1 and compares it to Molchanov's original hypothesis. It also explains how we used `Mathematica` to verify Conjecture 4.1 for the real reflection group H_3 and the Shephard groups G_{25}, G_{26}, G_{32}. We verify Conjecture 4.1 for rank 2 groups in Section 8. In Section 9, we use Theorem 1.1 to define the above q-analogues of the f-vector and h-vector, and we prove Theorem 1.5 giving a (q,t)-analogue of the transformation (1.4) that converts between f and h. We explain how specializations of these q-analogues give known product formulas for q-Catalan, q-Kirkman, and q-Narayana numbers in Section 10 and explain connections to graded *parking spaces*. Lastly, in Section 11, we compile the Hilbert series of $(S(V^*)\otimes\wedge V^*\wedge V)^W$ for all of the exceptional irreducible complex reflection groups.

2. Invariant theory of reflection groups

We begin by recounting some appearances of the *exponents* and *coexponents* in the invariant theory of reflection groups. Recall that a *reflection* on $V=\mathbb{C}^n$ is a linear transformation whose fixed point space is a hyperplane and a *reflection group* W is a subgroup of $\mathrm{GL}(V)$ generated by reflections. We assume all reflection groups are finite. Consequently, we may take an inner product on V with respect to which W acts by isometries and fix a basis of V so that the matrices giving the action are unitary. We write $\det=\det_V$ throughout for the determinant of elements of W acting on V. A reflection group is a *(finite) Coxeter group* or *real reflection group* if it is generated by reflections on $\mathbb{R}^n$, which then act on $\mathbb{C}^n$ by extension of scalars.

A large body of literature describes the invariant theory of reflection groups acting on $V=\mathbb{C}^n$ in terms of two sequences of positive integers, the *exponents* e_i and *coexponents* e_i^* of W,

$$e_1\le e_2\le\cdots\le e_n \quad\text{and}\quad e_1^*\le e_2^*\le\cdots\le e_n^*, \tag{2.1}$$

defined as follows. The dual action of W on V^* induces an action on the *symmetric algebra*

$$S(V^*)\cong\mathbb{C}[x_1,\dots,x_n]$$

where $x_1,\dots,x_n$ is the $\mathbb{C}$-basis for V^* dual to a $\mathbb{C}$-basis $y_1,\dots,y_n$ of V; the group W acts on $S(V^*)$ via invertible linear substitutions of the variables $x_1,\dots,x_n$. A theorem of Shephard and Todd [31] and of Chevalley [6] asserts that the W-invariant polynomials form a polynomial subalgebra:

$$S(V^*)^W=\mathbb{C}[f_1,\dots,f_n]$$

for some homogeneous f_i in $S(V^*)$ called *basic invariants*. Their polynomial degrees $d_1\le\cdots\le d_n$ are independent of the choice of the f_i. The exponents of W are then just the integers $e_i:=d_i-1$.

More generally, we may define U-exponents for any W-representation U by regarding the W-fixed space $(S(V^*)\otimes U)^W$ as a module over $S(V^*)^W$ via multiplication into the left tensor factor. This module is free of rank $\dim U$ by Chevalley's Theorem [6] (see [5, Prop. 4.3.3, eqn. (4.6)]) or by a result of Hochster and Eagon [15], and the *U-exponents* $e_1(U)\le\cdots\le e_{\dim U}(U)$ are the degrees of a homogeneous basis. Here, $S(V^*)\otimes U$ inherits the grading on $S(V^*)$ by polynomial degree. Note that these U-exponents are the degrees in which the representation U^* appears in the coinvariant algebra $S(V^*)/S(V^*)^W_+$.

As a special case, the V-exponents are the coexponents $e_i^*=e_i(V)$. In other words, $(S(V^*)\otimes V)^W$ is a free module over $S(V^*)^W$ and one may choose a basis $\{\theta_1,\dots,\theta_n\}$, called a set of *basic derivations*, with

$$\theta_i=\sum_{j=1}^n\theta_i^j\otimes y_j \quad\text{for homogeneous } \theta_i^j \text{ in } S(V^*) \text{ of degree } e_i^*\,. \tag{2.2}$$

When W is irreducible, there is a unique smallest coexponent $e_1^*=1$ corresponding to the *Euler derivation*, $\theta_1=\theta_E:=\sum_{i=1}^n x_i\otimes y_i$, which is always W-invariant (see [28], for example).

Solomon [35] considered the space of differential forms and showed that the *exterior algebra* $\wedge V^*$ tensored with $S(V^*)$ has W-fixed space which is an exterior algebra over $S(V^*)^W$ on exterior generators $\{df_1, \ldots, df_n\}$:

$$(S(V^*) \otimes \wedge V^*)^W = \bigwedge_{S(V^*)^W} \{df_1, \ldots, df_n\} \quad \text{where} \quad df := \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} \otimes x_i.$$

From this one can deduce that the exponents are alternatively defined as the V^*-exponents via $e_i := e_i(V^*)$. Orlik and Solomon [25, Thm. 3.1] generalized Solomon's Theorem, implying as a special case that the exterior algebra $\wedge V$ tensored with $S(V^*)$ has W-fixed space which is also an exterior algebra over base ring $S(V^*)^W$, this time with exterior generators given by the basic derivations $\{\theta_1, \ldots, \theta_n\}$ in $(S(V^*) \otimes V)^W$:

$$(S(V^*) \otimes \wedge V)^W = \bigwedge_{S(V^*)^W} \{\theta_1, \ldots, \theta_n\}\,. \tag{2.3}$$

Hilbert Series. The above structural results imply combinatorial descriptions for various *Hilbert series* $\mathrm{Hilb}(M, q) := \sum_{d \geq 0} \dim M_d \cdot q^d$ for graded (or doubly or triply graded) vector spaces $M = \bigoplus_{d \geq 0} M_d$. In each case, one compares the Hilbert series implied by variants on Molien's Theorem (see [37] or [5, Lemma 3.2.8]) to the expression implied by the above results on the structure of the various rings and modules. Appreviating $S = S(V^*)$, we observe the following.

- The Shephard-Todd-Chevalley Theorem on S^W implies that

$$\tfrac{1}{|W|} \sum_{w \in W} \frac{1}{\det(1 - qw)} = \mathrm{Hilb}(S^W, q) = \frac{1}{\prod_{i=1}^{n}(1 - q^{d_i})}\,. \tag{2.4}$$

- The definition of coexponents in terms of $(S \otimes V)^W$ implies that

$$\tfrac{1}{|W|} \sum_{w \in W} \frac{\chi_V(w^{-1})}{\det(1 - qw)} = \mathrm{Hilb}((S \otimes V)^W, q) = \Big(\sum_{i=1}^{n} q^{e_i^*}\Big) \frac{1}{\prod_{i=1}^{n}(1 - q^{d_i})}\,. \tag{2.5}$$

- Solomon's Theorem describing $(S \otimes \wedge V^*)^W$ implies this generalization of (2.4):

$$\tfrac{1}{|W|} \sum_{w \in W} \frac{\det(1 + tw)}{\det(1 - qw)} = \mathrm{Hilb}((S \otimes \wedge V^*)^W, q, t) = \prod_{i=1}^{n} \frac{1 + q^{e_i} t}{1 - q^{d_i}}\,, \tag{2.6}$$

  where the coefficient of $q^i t^k$ in the Hilbert series is the dimension of $(S^i \otimes \wedge^k V^*)^W$.
- The Orlik-Solomon Theorem describing $(S \otimes \wedge V)^W$ implies this different generalization of (2.4):

$$\tfrac{1}{|W|} \sum_{w \in W} \frac{\det(1 + sw^{-1})}{\det(1 - qw)} = \mathrm{Hilb}((S \otimes \wedge V)^W, q, t) = \prod_{i=1}^{n} \frac{1 + q^{e_i^*} s}{1 - q^{d_i}}\,, \tag{2.7}$$

  where the coefficient of $q^i s^r$ in the Hilbert series is the dimension of $(S^i \otimes \wedge^r V^*)^W$.

Duality groups. More recently, the first two authors [28] proved a structural statement in the invariant theory of *duality groups*, that is, reflection groups satisfying the *exponent-coexponent duality*

$$e_i + e^*_{n+1-i} = h := \max\{d_i\}.$$

For any reflection group W, the space $(S(V^*) \otimes \wedge V^* \otimes V)^W$ is a module over the $S(V^*)^W$-exterior algebra

$$(S(V^*) \otimes \wedge V^*)^W = \bigwedge_{S(V^*)^W} \{df_1, \ldots, df_n\}$$

via multiplication in the first two tensor factors. In general, it is *not free* as a module over this exterior algebra. But when W is a duality group, $(S(V^*) \otimes \wedge V^* \otimes V)^W$ is free as a module over the *subalgebra*

$$\bigwedge_{S(V^*)^W} \{df_1, \ldots, df_{n-1}\}$$

which omits the last exterior generator df_n. This similarly implies a combinatorial Hilbert series:

Theorem 2.8 ([28]). *For W a duality group,*

$$\begin{aligned}\tfrac{1}{|W|}\sum_{w\in W}\chi_V(w^{-1})\frac{\det(1+tw)}{\det(1-qw)} &= \operatorname{Hilb}\big((S(V^*)\otimes\wedge V^*\otimes V)^W,q,t\big)\\ &= \Big(\sum_{i=1}^{n} q^{e_i^*}\Big)\ \frac{(1+q^{-1}t)\ \prod_{i=1}^{n-1}(1+q^{e_i}t)}{\prod_{i=1}^{n}(1-q^{d_i})}\,.\end{aligned}\tag{2.9}$$

Note that setting $t=0$ in (2.9) gives (2.5) for duality groups W.

The W-invariants of $S(V^*)\otimes\wedge V^*\otimes\wedge^n V$. The following theorem holds for all complex reflection groups and agrees with extraction of the coefficient of s^n in Theorem 1.1. It follows from results of the second author [32, 33], but we include a proof since we have not seen it stated explicitly in the literature.

Theorem 2.10. *For any complex reflection group W acting on V,*

$$\tfrac{1}{|W|}\sum_{w\in W}\det(w^{-1})\frac{\det(1+tw)}{\det(1-qw)} = \operatorname{Hilb}\big((S(V^*)\otimes\wedge V^*\otimes\wedge^n V)^W,q,t\big) = \prod_{i=1}^{n}\frac{q^{e_i^*}+t}{1-q^{d_i}}\,.$$

Proof. Since $\wedge^n V$ carries the determinant character det of W, the S^W-module $(S\otimes\wedge V^*\otimes\wedge^n V)^W$ for $S=S(V^*)$ is the space of $\det^{-1}$-*relative invariants* for W acting on $S\otimes\wedge V^*$. It was shown in [32] (see also [34]) that the $\det^{-1}$-relative invariant 1-forms $(S\otimes V^*)^{\det^{-1}}$ form a free S^W-module and any basis $\{\omega_1,\ldots,\omega_n\}$ generates the $\det^{-1}$-relative invariant p-forms $(S\otimes\wedge^p V^*)^{\det^{-1}}$ freely over S^W with basis

$$\Big\{\tfrac{1}{Q^{p-1}}\cdot\omega_{i_1}\wedge\cdots\wedge\omega_{i_p} : 1\le i_1<\cdots<i_p\le n\Big\}.$$

Here $Q:=\prod_H \ell_H$ is the $\det^{-1}$-relative invariant in S of lowest degree, namely, the product of the linear forms ℓ_H whose vanishings define the reflecting hyperplanes H of W. Thus Q has degree equal to the number N^* of reflecting hyperplanes, with known formula (e.g., [5, §4.5.5, Remark 4.48])

$$\deg(Q)=:N^*=e_1^*+\cdots+e_n^*.$$

If $\omega_1,\ldots,\omega_n$ are homogeneous with polynomial degrees $m_1,\ldots,m_n$, (so each ω_i lies in $S^{m_i}\otimes V^*$), then

$$\begin{aligned}\frac{\operatorname{Hilb}\big((S\otimes\wedge V^*\otimes\wedge^n V)^W,q,t\big)}{\operatorname{Hilb}(S^W,q)} &= q^{\deg(Q)}\sum_{p=0}^{n}t^p\sum_{1\le i_1<\cdots<i_p\le n} q^{m_{i_1}+\cdots+m_{i_p}-p\deg(Q)}\\ &= q^{\deg(Q)}\prod_{i=1}^{n}(1+tq^{m_i-\deg(Q)}) = \prod_{i=1}^{n}(q^{e_i^*}+tq^{e_i^*+m_i-\deg(Q)}).\end{aligned}$$

Since $\operatorname{Hilb}(S^W,q)=\prod_{i=1}^n\frac{1}{1-q^{d_i}}$, the theorem will follow if we show that one can index so that

$$m_i=\deg(Q)-e_i^*=(e_1^*+\cdots+e_n^*)-e_i^*$$

for $i=1,2,\ldots,n$. To see this, we proceed as in the proof of [33, Cor. 4]. First note that the nondegenerate pairing $V\otimes\wedge^{n-1}V\to\wedge^n V$ is W-equivariant, where $\wedge^n V$ carries the character det. This implies $\wedge^{n-1}V\cong V^*\otimes\det$ as W-representations. Therefore the S^W-module of $\det^{-1}$-relative invariants in $S\otimes V^*$, which is isomorphic to the W-invariants $(S\otimes V^*\otimes\det)^W$, is also isomorphic to the W-invariants $\big(S\otimes\wedge^{n-1}V\big)^W$. However, by (2.3), the latter has S^W-basis $\{\theta_1\wedge\cdots\wedge\hat{\theta}_i\wedge\cdots\wedge\theta_n : i=1,2,\ldots,n\}$, whose elements indeed have degrees $(e_1^*+\cdots+e_n^*)-e_i^*$. ☐

Invariant derivation differential forms. Theorem 1.1 describes the (triply-graded) Hilbert series

$$\operatorname{Hilb}\big((S(V^*)\otimes\wedge V^*\otimes\wedge V)^W,q,t,s\big)=\tfrac{1}{|W|}\sum_{r=0}^{n}\sum_{w\in W}\frac{\det(1+tw)\det(1+sw^{-1})}{\det(1-qw)},\tag{2.11}$$

where the coefficient of $q^it^ks^r$ is the dimension of $(S^i(V^*)\otimes\wedge^k V^*\otimes\wedge^r V)^W$, in terms of exponents and coexponents and specializes to all of (2.4), (2.5), (2.6), (2.7), (2.9) and Theorem 2.10. However, it applies only to the subfamily of *coincidental* reflection groups, described further here.

Coincidental groups. A reflection group W is *coincidental* if it is an irreducible duality group whose exponents (or equivalently, its degrees, or coexponents, or codegrees) form an arithmetic sequence. The coincidental reflection groups comprise (using notation from the classification of Shephard and Todd [31])

- the real reflection groups A_n, B_n/C_n, $I_2(m)$, H_3,
- the monomial groups $G(d,1,n)$,
- all rank 2 duality groups, namely, G_4, G_5, G_6, G_8, G_9, G_{10}, G_{14}, G_{16}, G_{17}, G_{18}, G_{20}, and G_{21}, and
- the groups G_{25}, G_{26}, and G_{32}.

Note that coincidental groups include all Shephard groups that are not Coxeter groups. Among the Coxeter-Shephard groups, coincidental groups exclude type D_n for $n \geq 4$ and the real exceptional groups, E_6, E_7, E_8, F_4, H_4, i.e., those groups whose Coxeter diagram contains one of D_4, F_4, or H_4 as a subdiagram. The coincidental groups have made multiple appearances recently, for example, in the work of Miller [22, Thm. 14], [23, Thm. 2]. See [13, §5] for examples of real coincidental types in the literature.

Numerology of coincidental groups is governed by two parameters, the *smallest exponent* e_1 and the *gap*

$$a := d_i - d_{i-1} = e_i - e_{i-1} = e_i^* - e_{i-1}^*$$

between any two successive exponents, or fundamental degrees, or coexponents:

$$\begin{array}{rcllllll}
(d_1, d_2, \ldots, d_n) & = & (& e_1+1, & e_1+1+a, & e_1+1+2a & , \ldots, & e_1+1+a(n-1) \quad) \\
(e_1, e_2, \ldots, e_n) & = & (& e_1, & e_1+a, & e_1+2a & , \ldots, & e_1+a(n-1) \quad) \\
(e_1^*, e_2^*, \ldots, e_n^*) & = & (& 1, & 1+a, & 1+2a & , \ldots, & 1+a(n-1) \quad).
\end{array}$$

3. Type A and the monomial groups

We begin compiling our verification of Theorem 1.1 with the Weyl groups of type A and the infinite family of Shephard-Todd groups $G(d,1,n)$.

Recall that for positive integers d,n, the complex reflection group $G(d,1,n)$ is the set of all $n \times n$ matrices in $\mathrm{GL}(V) = \mathrm{GL}_n(\mathbb{C})$ that are *monomial* (exactly one nonzero entry in each row and column) with nonzero entries all d^{th} roots-of-unity in $\mathbb{C}$. Any element w in $G(d,1,n)$ maps the basis vectors $y_1, \ldots, y_n$ of $V = \mathbb{C}^n$ to $\zeta^{m_1} y_{\pi(1)}, \ldots, \zeta^{m_n} y_{\pi(n)}$ for some permutation $\pi = \pi(w)$ in the symmetric group $\mathfrak{S}_n = G(1,1,n)$, where ζ is the complex root-of-unity $e^{\frac{2\pi i}{d}}$. In fact, $G(d,1,n) = \mathfrak{S}_n \ltimes (\mathbb{Z}/d\mathbb{Z})^n$ since the map $w \mapsto \pi(w)$ is a surjective group homomorphism, $G(d,1,n) \longrightarrow \mathfrak{S}_n$, whose kernel is the subgroup $(\mathbb{Z}/d\mathbb{Z})^n$ of diagonal matrices within $G(d,1,n)$. We need to draw a distinction between the symmetric group $\mathfrak{S}_n$ acting as the permutation matrices $G(1,1,n)$ on one hand and acting as the Weyl group A_{n-1} on the other hand:

- The group $G(1,1,n)$ acts on $V = \mathbb{C}^n$ *reducibly*, with fixed space $V^{\mathfrak{S}_n} = \mathbb{C}(y_1 + \cdots + y_n)$.
- The group $W(A_{n-1})$ acts *irreducibly* on the quotient space $V/\mathbb{C}(y_1 + \cdots + y_n) \cong \mathbb{C}^{n-1}$.

For any finite group W, we introduce a shorthand notation for the Hilbert series describing the isotypic component in $S(V^*) \otimes \wedge V^*$ corresponding to a W-representation M with character χ. Again, we use a Molien type theorem to write this Hilbert series as a sum over group elements and abbreviate $S = S(V^*)$. We are interested in the special case when χ is the character of the W-representation $\wedge^r V$ giving (2.11).

Definition 3.1. For any finite subgroup W of $\mathrm{GL}(V)$ and any character χ of a W-module M, define

$$P_W(\chi; q, t) := \mathrm{Hilb}((S \otimes \wedge V^* \otimes M^*)^W, q, t) = \tfrac{1}{|W|} \sum_{w \in W} \chi(w) \frac{\det(1+tw)}{\det(1-qw)} \quad \text{and}$$

$$\chi_{S \otimes \wedge V^*}(q,t)(w) := \sum_{j=0}^{\infty} \sum_{k=0}^{n} q^j \, t^k \, \chi_{S^j \otimes \wedge^k V^*}(w) \quad \text{for } w \text{ in } W.$$

The second expression, $\chi_{S \otimes \wedge V^*}(q,t)(w)$, is a class function on W with values in the ring $\mathbb{Z}[t][[q]]$. Notice that in terms of the usual inner product $\langle \cdot, \cdot \rangle_W$ on W-class functions,

$$P_W(\chi; q, t) = \langle \, \chi \, , \, \chi_{S \otimes \wedge V^*}(q,t) \, \rangle_W \, .$$

The next two subsections review formulas for $P_W(\chi; q, t)$ for χ a W-irreducible character for $W = W(A_{n-1})$ and $G(d,1,n)$. Since $\wedge^r V$ and $(\wedge^r V)^*$ are irreducible W-representations (see [18, §24-3]), we examine

$$P_W((\chi_{\wedge^r V})^*; q, t) = \mathrm{Hilb}((S \otimes \wedge V^* \otimes \wedge^r V)^W, q, t) \tag{3.2}$$

to verify Theorem 1.1 for the coincidental types $W(A_{n-1})$ and $G(d,1,n)$ with $d \geq 2$.

The Type A formula of Kirillov-Pak, Molchanov, Thibon, Gyoja-Nishiyama-Shimura. The following "hook-content" formula for $P_W(\chi; q, t)$ when $W = \mathfrak{S}_n = G(1,1,n)$ is the building block for the rest. This formula was proven first by Kirillov and Pak [19, eqn. (4)] bijectively and then more algebraically by Molchanov [24, eqn. (2)]. It was also deduced from symmetric function identities by Thibon [39, Thm. 4.3] and by Gyoja, Nishiyama, and Shimura [14, eqn. (3.2)]. To state the formula, recall that the irreducible characters of the symmetric group $\mathfrak{S}_n$ can be indexed as $\{\chi^\lambda\}$ where $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq 0)$ runs through the partitions of n, that is, $|\lambda| := \sum_i \lambda_i = n$. Define $n(\lambda) := \sum_{i \geq 1}(i-1)\lambda_i$. Recall also the notion of the *Ferrers diagram* for λ, containing the *cells* $x = (i,j)$ in row i and column j for $1 \leq j \leq \lambda_i$. The cell $x = (i,j)$ is said to have *content* $c(x) := j - i$ and *hooklength* $h(x) := \lambda_i + \#\{k : \lambda_k \geq i\} - j$.

Theorem 3.3. [14, 19, 24] *For $W = \mathfrak{S}_n = G(1,1,n)$ acting on $V = \mathbb{C}^n$ and any character χ^λ of W,*

$$P_{\mathfrak{S}_n}(\chi^\lambda; q, t) = q^{n(\lambda)} \prod_{x \in \lambda} \frac{1 + t\, q^{c(x)}}{1 - q^{h(x)}}.$$

The theorem gives the following corollary for the irreducible action A_{n-1} of the symmetric group $\mathfrak{S}_n$.

Corollary 3.4. *For the Weyl group $W(A_{n-1})$ acting on $V = \mathbb{C}^{n-1}$ and any irreducible character χ^λ,*

$$P_{W(A_{n-1})}(\chi^\lambda; q, t) = \frac{1-q}{1+t} \;\; P_{\mathfrak{S}_n}(\chi^\lambda; q, t) \tag{3.5}$$

$$= \frac{1-q}{1+t}\; q^{n(\lambda)} \prod_{x \in \lambda} \frac{1 + t\, q^{c(x)}}{1 - q^{h(x)}}. \tag{3.6}$$

In particular, for χ^λ the character $\chi_{\wedge^r V}$ of the W-representation $\wedge^r V$ for some fixed $r = 0, \ldots, n$,

$$P_{W(A_{n-1})}(\chi_{\wedge^r V}; q, t) = q^{r + \binom{r+1}{2}} \begin{bmatrix} n-1 \\ r \end{bmatrix}_q \frac{(-t\,q; q)_{n-r-1}(-t\,q^{-1}; q^{-1})_r}{(q;q)_{n-1}}, \tag{3.7}$$

and the assertion of Theorem 1.1 holds for type A_{n-1}.

Proof. Equation (3.5) follows from the fact that a permutation matrix π in $G(1,1,n)$ acts on $\mathbb{C}^n$ with one extra eigenvalue $+1$ compared to its action w on $\mathbb{C}^{n-1}$ as an element in $W(A_{n-1})$, and hence

$$\frac{\det(1+t\pi)}{\det(1-q\pi)} = \frac{1+t}{1-q} \cdot \frac{\det(1+tw)}{\det(1-qw)}.$$

Theorem 3.3 then gives (3.6). For (3.7), note that $(\chi_{\wedge^r V})^* = \chi_{\wedge^r V} = \chi^\lambda$ for $\lambda = (n-r, 1^r)$, which has

$$n(\lambda) = \binom{r+1}{2},$$
$$\text{cell contents } c(x) = (0, 1, 2, \ldots, n-1-r, -1, -2, \ldots, -r),$$
$$\text{hooklengths } h(x) = (1, 2, \ldots, n, 1, 2, \ldots, r).$$

We apply (3.6) in this special case and obtain

$$\begin{aligned} P_{W(A_{n-1})}((\chi_{\wedge^r V})^*; q, t) = P_{W(A_{n-1})}(\chi_{\wedge^r V}; q, t) &= \frac{1-q}{1+t} \;\; q^{n(\lambda)} \prod_{x \in \lambda} \frac{1 + tq^{c(x)}}{1 - q^{h(x)}} \\ &= \frac{1-q}{1+t}\; q^{\binom{r+1}{2}} \frac{1+t}{1-q^n} \prod_{i=1}^{n-r-1} \frac{1+tq^i}{1-q^i} \prod_{i=1}^{r} \frac{1+tq^{-i}}{1-q^i} \\ &= q^{\binom{r+1}{2}} \; \frac{1-q}{1-q^n} \; \frac{(-tq;q)_{n-r-1}}{(q;q)_{n-r-1}} \; \frac{(-tq^{-1};q^{-1})_r}{(q;q)_r} \\ &= q^{\binom{r+1}{2}} \; \frac{1-q}{1-q^n} \; \begin{bmatrix} n-1 \\ r \end{bmatrix}_q \; \frac{(-tq;q)_{n-r-1}\;(-tq^{-1};q^{-1})_r}{(q;q)_{n-1}} \\ &= q^{r+\binom{r}{2}} \; \begin{bmatrix} n-1 \\ r \end{bmatrix}_q \; \frac{(-tq;q)_{n-r-1}\;(-tq^{-1};q^{-1})_r}{(q^2;q)_{n-1}} \end{aligned} \tag{3.8}$$

using (1.2) in the penultimate equality.

Finally, to verify Theorem 1.1 for $W(A_{n-1})$, we apply (3.2) and check that the last expression in (3.8) agrees with the expression in Theorem 1.1′ (see (2.11)). This holds because $W(A_{n-1})$ has rank $n-1$ and exponents

$$(e_1,\ldots,e_{n-1}) = (1,2,\ldots,n-1),$$

so that its smallest exponent is $e_1 = 1$ and the gap between exponents is $a = 1$. We also use the fact that $W(A_{n-1})$ is a real reflection group, so that $V \cong V^*$ and $e_i^* = e_i$. $\square$

Koike's formula for $G(d,1,n)$. We review here a formula of Koike [20] generalizing to the monomial groups $W = G(d,1,n)$ for $d \geq 2$ the calculation of $P_W(\chi,t)$ completed for $G(2,1,n) = W(B_n)$ by Kirillov and Pak [19, eqn. (6)]; see also Gyoja, Nishiyama, and Shimura [14, eqn. (3.9)] for the case of $W(B_n)$.

Fix $d \geq 2$, and let us abbreviate $W_n := G(d,1,n)$ acting on $V = \mathbb{C}^n$. The irreducible characters of W_n can be indexed by *d-multipartitions of n*

$$\underline{\lambda} = (\lambda^{(0)}, \lambda^{(1)}, \ldots, \lambda^{(d-1)}) \tag{3.9}$$

in which each $\lambda^{(i)}$ is a partition of n_i and $\sum_{i=0}^{d-1} n_i = n$. Denoting $\underline{n} := (n_0, n_1, \ldots, n_{d_1})$, let $W_{\underline{n}}$ be the subgroup of W_n isomorphic to $W_{n_0} \times W_{n_1} \times \cdots \times W_{n_{d-1}}$ consisting of the block diagonal matrices in W_n with diagonal block sizes specified by $\underline{n}$.

Recall that any element w in $G(d,1,n)$ can be written uniquely as $w = \mathrm{diag}(w) \cdot \pi(w)$, the product of a diagonal matrix $\mathrm{diag}(w)$ and a permutation matrix $\pi(w)$ in $G(1,1,n)$. This gives rise to a 1-dimensional character recording the determinant of the diagonal part of w (i.e., the product of the nonzero entries in w):

$$\epsilon : W_n \longrightarrow \mathbb{C}^\times, \qquad w \mapsto \det(\mathrm{diag}(w))\,. \tag{3.10}$$

Given any $\mathfrak{S}_n$-character χ, one can *inflate* it along π to a W_n-character that we will denote $\chi \Uparrow_{\mathfrak{S}_n}^{W_n}$. Given any $W_{\underline{n}}$-character χ, one can *induce* it up to W_n, giving a character that we will denote $\chi \uparrow_{W_{\underline{n}}}^{W_n}$. One then has the following description for the irreducible W_n-character indexed by $\underline{\lambda}$ as in (3.9):

$$\chi^{\underline{\lambda}} = \left(\bigotimes_{i=0}^{d-1} \epsilon^i \otimes \left(\chi^{\lambda_i} \Uparrow_{\mathfrak{S}_{n_i}}^{W_{n_i}} \right) \right) \uparrow_{W_{\underline{n}}}^{W_n}.$$

Here ϵ is the same degree 1 character of W_n restricted to each W_{n_i}, and ϵ^i is its i^{th} tensor power.

We can now state Koike's result. We include a proof which is shorter and less computational than the one in [20, pp. 545-548], following the methodology of Kirillov and Pak.

Theorem 3.11. [20, Theorem 1] *For any d-multipartition $\underline{\lambda}$ of n as above,*

$$P_{W_n}(\chi^{\underline{\lambda}}; q, t) = P_{\mathfrak{S}_{n_0}}(\chi^{\lambda^{(0)}}; q^d, tq^{d-1}) \cdot \prod_{i=1}^{d-1} q^{n_i(d-i)} \cdot P_{\mathfrak{S}_{n_i}}(\chi^{\lambda^{(i)}}; q^d, tq^{-1}).$$

Proof. (cf. [19, Proof of Lemma 1]) By Frobenius reciprocity,

$$P_{W_n}(\chi^{\underline{\lambda}}; q, t) = \left\langle\, \chi^{\underline{\lambda}},\, \chi_{S\otimes\wedge V^*}(q,t) \,\right\rangle_{W_n} = \prod_{i=0}^{d-1} \left\langle\, \epsilon^i \otimes \left(\chi^{\lambda^{(i)}} \Uparrow_{\mathfrak{S}_{n_i}}^{W_{n_i}} \right),\, \chi_{S\otimes\wedge V^*}(q,t) \,\right\rangle_{W_{n_i}}. \tag{3.12}$$

It remains to compute the i^{th} factor in the product. For ease of notation, replace n_i by n and $\lambda^{(i)}$ by λ, so that we may rewrite the i^{th} factor simply as

$$\begin{aligned} \left\langle\, \epsilon^i \otimes \left(\chi^{\lambda} \Uparrow_{\mathfrak{S}_n}^{W_n} \right),\, \chi_{S\otimes\wedge V^*}(q,t) \,\right\rangle_{W_n} &= \left\langle\, \chi^{\lambda} \Uparrow_{\mathfrak{S}_n}^{W_n},\, \epsilon^{-i} \otimes \chi_{S(V^*)\otimes\wedge V^*}(q,t) \,\right\rangle_{W_n} \\ &= \left\langle\, \chi^{\lambda},\, \chi_{(\epsilon^{-i}\otimes S(V^*)\otimes\wedge V^*)^{(\mathbb{Z}/d\mathbb{Z})^n}}(q,t) \,\right\rangle_{\mathfrak{S}_n}. \end{aligned} \tag{3.13}$$

Here the last equality uses a general adjointness statement (see, e.g., [11, §4.1.6]) for quotient groups $G \to G/N$, taking $(G, N, G/N)$ to be $(W_n, (\mathbb{Z}/d\mathbb{Z})^n, \mathfrak{S}_n)$: For any G-representation A, with residual G/N-representation on its N-fixed space A^N, and any G/N-representation B,

$$\langle\, \chi_B \Uparrow_{G/N}^{G},\, \chi_A \,\rangle_G = \langle\, \chi_B,\, \chi_{A^N} \,\rangle_{G/N}\,.$$

We next examine, for $0 \leq i \leq d-1$, the $(\mathbb{Z}/d\mathbb{Z})^n$-fixed space,

$$\left(\epsilon^{-i} \otimes S(V^*) \otimes \wedge V^*\right)^{(\mathbb{Z}/d\mathbb{Z})^n} . \tag{3.14}$$

Note the following tensor product decomposition, compatible with the $(\mathbb{Z}/d\mathbb{Z})^n$-action:

$$S(V^*) \otimes \wedge V^* = \bigotimes_{j=1}^{n} \mathbb{C}[x_j] \otimes \wedge \mathbb{C}x_j .$$

Since the character ϵ restricted to $(\mathbb{Z}/d\mathbb{Z})^n$ is just the tensor product of ϵ on each of the factors $\mathbb{Z}/d\mathbb{Z}$, the $(\mathbb{Z}/d\mathbb{Z})^n$-fixed space (3.14) is the tensor product of these $\mathbb{Z}/d\mathbb{Z}$-fixed spaces:

$$\left(\epsilon^{-i} \otimes \mathbb{C}[x_j] \otimes \wedge \mathbb{C}x_j\right)^{\mathbb{Z}/d\mathbb{Z}} = \begin{cases} \left(\mathbb{C}[x_j^d] \otimes 1\right) \oplus \left(x_j^{d-1}\mathbb{C}[x_j^d] \otimes x_j\right) & \text{if } i = 0, \\ \left(x_j^{d-i}\mathbb{C}[x_j^d] \otimes 1\right) \oplus \left(x_j^{d-1-i}\mathbb{C}[x_j^d] \otimes x_j\right) & \text{if } 1 \leq i \leq d-1. \end{cases}$$

The $i = 0$ case. The space $(\mathbb{C}[x_j^d] \otimes 1) \oplus (x_j^{d-1}\mathbb{C}[x_j^d] \otimes x_j)$ is the tensor product of a symmetric algebra $\mathbb{C}[x_j^d]$ on the generator x_j^d with an exterior algebra on the generator $x_j^{d-1} \otimes x_j$. Consequently, the $(\mathbb{Z}/d\mathbb{Z})^n$-fixed space (3.14) is a symmetric algebra $\mathbb{C}[x_1^d, \ldots, x_n^d]$ tensored with the exterior algebra on the generators $\{x_j^{d-1} \otimes x_j\}_{j=1}^n$. Since the residual action of $\mathfrak{S}_n$ permutes the subscripts as usual, one concludes that the last expression in (3.13) is $P_{\mathfrak{S}_n}(\chi^\lambda; q^d, tq^{d-1})$ for $i = 0$.

The $1 \leq i \leq d-1$ case. We proceed as in the $i = 0$ case. First enlarge coefficients from $\mathbb{C}[x_i]$ to $\mathrm{Frac}(\mathbb{C}[x_i])$. Then $(x_j^{d-i}\mathbb{C}[x_j^d] \otimes 1) \oplus (x_j^{d-1-i}\mathbb{C}[x_j^d] \otimes x_j)$ is a free module of rank 1 with basis element x_j^{d-i} over the subalgebra of $\mathrm{Frac}(\mathbb{C}[x_i]) \otimes \wedge \mathbb{C}x_j$ which is the tensor product of a symmetric algebra $\mathbb{C}[x_j^d]$ on generator x_j^d with an exterior algebra on generator $x_j^{-1} \otimes x_j$. Consequently, the $(\mathbb{Z}/d\mathbb{Z})^n$-fixed space (3.14) is a free module of rank 1 with basis element $(x_1 \cdots x_n)^{d-i}$ over the subalgebra of $\mathrm{Frac}(S) \otimes \wedge V^*$ given as the tensor product of the symmetric algebra $\mathbb{C}[x_1^d, \ldots, x_n^d]$ with the exterior algebra on the generators $\{x_j^{-1} \otimes x_j\}_{j=1}^n$. Since $\mathfrak{S}_n$ still permutes the subscripts, the last expression in (3.13) is just $q^{n(d-i)} P_{\mathfrak{S}_n}(\chi^\lambda; q^d, tq^{-1})$.

We substitute these expressions for each i in (3.12) to obtain the product on the right in the proposition. $\square$

Remark 3.15. For the sake of the reader wishing to compare notation with that of Koike [20], note that the group we call $W_n = G(d, 1, n)$ is his group $G_{n,d}$, and the group that we call $G(de, e, n)$ below is his group $G_{n,d,e}$. Also, because he works with $S(V) \otimes \wedge(V)$ rather than $S(V^*) \otimes \wedge(V^*)$, his Theorem 1 calculates what we denote here by $P_{W_n}((\chi^{\underline{\lambda}})^*; q, t)$, where one can identify the contragredient representation of $\chi^{\underline{\lambda}} = \chi^{(\lambda^{(0)}, \lambda^{(1)}, \ldots, \lambda^{(r-1)})}$ with $(\chi^{\underline{\lambda}})^* = \chi^{(\lambda^{(0)}, \lambda^{(r-1)}, \ldots, \lambda^{(2)}, \lambda^{(1)})}$.

Corollary 3.16. *The conclusion of Theorem 1.1 holds for the Shephard-Todd family $G(d, 1, n)$ with $d \geq 2$.*

Proof. According to (3.2), we should compute $P_W((\chi_{\wedge^r V})^*; q, t)$ for $W = G(d, 1, n)$ acting on $V = \mathbb{C}^n$ and $0 \leq r \leq n$. One can check that $(\chi_{\wedge^r V})^*$ is the W-irreducible character $\chi^{\underline{\lambda}}$ for $\underline{\lambda} = (\lambda^{(0)}, \ldots, \lambda^{(d-1)})$ where

$$\lambda^{(0)} = (n-r), \quad \lambda^{(1)} = \lambda^{(2)} = \cdots = \lambda^{(d-2)} = \varnothing, \quad \text{and} \quad \lambda^{(d-1)} = (1^r).$$

Hence Theorem 3.11 together with Corollary 3.4 gives

$$\begin{aligned} P_{G(d,1,n)}((\chi_{\wedge^r V})^*; q, t) &= P_{\mathfrak{S}_{n-r}}(\chi^{(n-r)}; q^d, tq^{d-1}) \; q^r \; P_{\mathfrak{S}_r}(\chi^{(1^r)}; q^d, tq^{-1}) \\ &= \prod_{x \in (n-r)} \frac{1 + tq^{d-1+dc(x)}}{1 - q^{dh(x)}} \; q^r \; q^{d\binom{r}{2}} \prod_{x \in (1^r)} \frac{1 + tq^{-1+dc(x)}}{1 - q^{dh(x)}} \\ &= q^{r + d\binom{r}{2}} \prod_{i=1}^{n-r} \frac{1 + tq^{d-1+d(i-1)}}{1 - q^{di}} \prod_{i=1}^{r} \frac{1 + tq^{-1-d(i-1)}}{1 - q^{di}} \\ &= q^{r + d\binom{r}{2}} \begin{bmatrix} n \\ r \end{bmatrix}_{q^d} \frac{(-tq^{d-1}; q^d)_{n-r} \, (-tq^{-1}; q^{-d})_n}{(q^d; q^d)_n} \end{aligned}$$

using (1.2) and (1.3) in the second and third equalities. As $G(d,1,n)$ has rank n and exponents

$$(e_1,\ldots,e_n) = (d-1, 2d-1, \ldots, nd-1),$$

its smallest exponent is $e_1 = d-1$ and the gap between exponents is $a = d$. Hence this last expression agrees with that in Theorem 1.1′, which is equivalent to Theorem 1.1. $\square$

Koike's formula for $G(de,e,n)$. Koike also generalized Theorem 3.11 from the wreath product groups $G(d,1,n)$ to the entire infinite family $G(de,e,n)$ in Shephard and Todd's classification of irreducible complex reflection groups [31]. Here $G(de,e,n)$ for positive integers d,e,n is the the kernel of the degree one character $\epsilon^d : G(de,1,n) \to \mathbb{C}^\times$, where ϵ was defined in (3.10). In other words, $G(de,e,n)$ is the group of monomial $n\times n$ matrices whose nonzero entries are all $(de)^{th}$ roots-of-unity and for which the product of the nonzero entries is a d^{th} root-of-unity. Although the group $G(de,e,n)$ is only coincidental when $e=1$, we find it worthwhile to state his result and then use it to compute the Hilbert series of $(S\otimes\wedge V^*\otimes\wedge^r V)^{G(de,e,n)}$, so that we can see how it would differ from the form in Theorem 1.1 when $e\geq 2$.

First we recall the parametrization of irreducible $G(de,e,n)$-representations; see, e.g., [20, §2]. Given an irreducible $G(de,1,n)$-character $\chi^{\underline{\lambda}}$ corresponding to a multipartition $\underline{\lambda} = (\lambda^{(0)},\lambda^{(1)},\ldots,\lambda^{(de-1)})$ of n, consider the superscripts i in $\lambda^{(i)}$ as taken modulo de, so that $\lambda^{(i+de)} = \lambda^{(i)}$ for all integers i. Then the restriction of $\chi^{\underline{\lambda}}$ from $G(de,1,n)$ to $G(de,e,n)$ depends only upon the orbit of $\underline{\lambda}$ under the operation that replaces $\lambda^{(i)}$ by $\lambda^{(i+d)}$ for all i. If one fixes a representative $\underline{\lambda}$ of this orbit and defines a positive integer

$$\mu(\underline{\lambda}) := \min\{m\geq 1 : \lambda^{(i)} = \lambda^{(i+dm)} \text{ for all } i\},$$

then $\chi^{\underline{\lambda}}$ decomposes upon restriction to $G(de,e,n)$ into $e/\mu(\underline{\lambda})$ inequivalent $G(de,e,n)$-irreducible characters, and each $G(de,e,n)$-irreducible arises once in this way. Koike's result may then be stated as follows.

Theorem 3.17. [20, Theorem 2] *For a multipartition $\underline{\lambda}$ of n parametrizing an irreducible character $\chi^{\underline{\lambda}}$ of $G(de,1,n)$, let $\hat{\chi}^{\underline{\lambda}}$ denote any $G(de,e,n)$-irreducible constituent of the restriction to $G(de,e,n)$. Then*

$$P_{G(de,e,n)}(\hat{\chi}^{\underline{\lambda}};q,t) = \sum_{v=0}^{\mu(\underline{\lambda})-1} P_{\mathfrak{S}_{n_{dv}}}(\chi^{\lambda^{(dv)}};q^{de},tq^{de-1})\cdot\prod_{i=1}^{de-1} q^{n_{dv+i}\cdot(de-i)}\cdot P_{\mathfrak{S}_{n_{i+dv}}}(\chi^{\lambda^{(i+dv)}};q^{de},tq^{-1}).$$

In particular, the answer is independent of the chosen irreducible constituent $\hat{\chi}^{\underline{\lambda}}$ and depends only on $\chi^{\underline{\lambda}}$.

Note that for $e=1$, every $\underline{\lambda}$ has $\mu(\underline{\lambda})=1$, so that the sum has only the $v=0$ term, recovering Theorem 3.11.

Corollary 3.18. *Let $W = G(de,e,n)$ with $d,e\geq 1$ and $n,de\geq 2$. Then the various Hilbert series*

$$\mathrm{Hilb}((S\otimes\wedge V^*\otimes\wedge^r V)^W,q,t) = P_W((\chi_{\wedge^r V})^*;q,t)$$

for $0\leq r\leq n$ have these formulas:

- *When $r=0$, it is*
$$\frac{(-tq^{de-1};q^{de})_{n-1}(1+tq^{dn-1})}{(q^{de};q^{de})_{n-1}(1-q^{dn})}.$$
- *When $1\leq r\leq n-1$ and $d\geq 2$, it is*
$$q^{de\binom{r}{2}+r}\,\frac{(-tq^{-1};q^{-de})_r(-tq^{de-1};q^{de})_{n-1-r}}{(q^{de};q^{de})_r(q^{de};q^{de})_{n-r}(1-q^{dn})}\left(1-q^{den}+tq^{-1}\left(q^{de(n-r)}(1-q^{dn})+q^{dn}-q^{den}\right)\right).$$
- *When $1\leq r\leq n-1$ and $d=1$, it is*
$$\begin{aligned} q^{e\binom{r}{2}}\,\frac{(-tq^{-1};q^{-e})_{r-1}(-tq^{e-1};q^e)_{n-1-r}}{(q^e;q^e)_r(q^e;q^e)_{n-r}(1-q^n)} \Big(& q^r(1+tq^{(n-r)e-1})(1+tq^{-(r-1)e-1})(1-q^n) \\ &+ q^{(n-r)(e-1)-1}(1+tq^{e-1})(1-q^n)(q+t) \\ &+ q^{r-1}(1+tq^{-(r-1)e-1})(q+t)(q^n-q^{n(e-1)})\Big). \end{aligned}$$
- *When $r=n$, it is*
$$q^{de\binom{n-1}{2}+n-1}\,\frac{(-tq^{-1};q^{-de})_{n-1}}{(q^{de};q^{de})_{n-1}(1-q^{dn})}\cdot\begin{cases}(q^{de(n-1)+1}+t) & \text{if } d\geq 2,\\ (q^{(n-1)(e-1)}+t) & \text{if } d=1.\end{cases}$$

Proof. We only sketch the somewhat tedious proofs, which are of the same nature as those for Corollary 3.16.

The $G(de,e,n)$-representation $(\wedge^r V)^*$ is the restriction of the $G(de,1,n)$-representation $(\wedge^r V)^*$, and the latter has character $\chi^{\underline{\lambda}}$ where $\underline{\lambda} = ((n-r), \varnothing, \varnothing, \dots, \varnothing, (1^r))$. Thus one can apply Theorem 3.17 to compute $P_{G(de,e,n)}(\hat{\chi}^{\underline{\lambda}}; q, t)$. In this case, $\mu(\underline{\lambda}) = e$, and so the sum in Theorem 3.17 always has e terms.

When $r = 0$, since $\underline{\lambda}$ has only one nonempty component $\lambda^{(0)} = (n)$, each summand in Theorem 3.17 for $v = 0, 1, 2, \dots, e-1$ has only one non-unit factor: the $v = 0$ summand is the factor $P_{\mathfrak{S}_n}(\chi^{(n)}; q^{de}, tq^{de-1})$, and each of the $v = 1, 2, \dots, e-1$ summands is the factor indexed by $i = d(e-v)$ in the product. After pulling out common factors, one can sum the geometric series over $v = 1, 2, \dots, e-1$, and some simplification then leads to the formula given in the corollary. The proof for $r = n$ is extremely similar.

When $1 \le r \le n-1$, since $\underline{\lambda}$ has two nonempty components $\lambda^{(0)} = (n-r)$ and $\lambda^{(de-1)} = (1^r)$, each summand for Theorem 3.17 for $v = 0, 1, 2, \dots, e-1$ is a product of two non-unit factors.

- The $v = 0$ summand is the factor in front of the product times the $i = 1$ factor within the product.
- If $d = 1$, the $v = 1$ summand is similarly the factor in front times the factor indexed by $i = e-1$ within the product, while each of the $v = 2, 3, \dots, e-1$ summands is the product of the two factors for $i = e-v, e-v+1$ in the product.
- If $d \ge 2$ so that $e - 1 < de - 1$, each of the $v = 1, 2, \dots, e-1$ summands is the product of the two factors for $i = d(e-v), d(e-v)+1$ in the product.

In each case, one pulls out common factors, sums a geometric series, and simplifies to obtain the formula. □

Remark 3.19. One can compare the formulas for the Hilbert series in Corollary 3.18 to see how they differ from the formulas in Theorem 1.1. Note that $G(de,e,n)$ has

$$\begin{aligned}
&(d_1, d_2, \dots, d_n) = (de, 2de, \dots, (n-1)de, dn)\,,\\
&(e_1, e_2, \dots, e_n) = (de-1, 2de-1, \dots, (n-1)de-1, dn-1)\,,\\
&(e_1^*, e_2^*, \dots, e_n^*) = \begin{cases} (1, 1+de, 1+2de, \dots, 1+(n-2)de, 1+(n-1)de) & \text{if } d \ge 2,\\ (1, 1+e, 1+2e, \dots, 1+(n-2)e, (n-1)(e-1)) & \text{if } d = 1\,. \end{cases}
\end{aligned} \tag{3.20}$$

It is then not hard to check that the cases $r = 0$ and $r = n$ of Corollary 3.18 agree with the formula given in Theorem 1.1, as predicted by Solomon's formula (2.6) for $r = 0$ and Theorem 2.10 for $r = n$.

For $1 \le r \le n-1$, one can check using (3.20) that the formula in Theorem 1.1 would assert for $d \ge 2$ that

$$\mathrm{Hilb}((S \otimes \wedge V^* \otimes \wedge^r V)^W, q, t) = q^{de\binom{r}{2}+r} \frac{(-tq^{-1}; q^{-de})_r (-tq^{de-1}; q^{de})_{n-1-r} (1 + tq^{d\min\{(n-r)e, n\}-1})}{(q^{de}; q^{de})_r (q^{de}; q^{de})_{n-r}}\,.$$

In general, this formula is *incorrect*—one can check that it disagrees with the corresponding expression in Corollary 3.18 except when $e = 1$, namely when $G(de,e,n) = G(d,1,n)$, a coincidental group.

Similarly, for $d = 1$, so that $W = G(e,e,n)$, one can check using (3.20) that the formula in Theorem 1.1 would assert that

$$\begin{aligned}
&\mathrm{Hilb}((S \otimes \wedge V^* \otimes \wedge^r V)^W, q, t)\\
&= q^{e\binom{r}{2}+r} \cdot \frac{(-tq^{-1}; q^{-e})_{r-1} (-tq^{e-1}; q^e)_{n-1-r}}{(q^e; q^e)_{r-1} (q^e; q^e)_{n-r}}\\
&\qquad \cdot \frac{(1 + tq^{\min\{(n-r)e, n\}-1})(1 + tq^{-\min\{1+(r-1)e, (n-1)(e-1)\}})}{(1 - q^{re})(1 - q^n)} \cdot \left(1 - q^{ne-re} + q^{ne-re-n} - q^{ne-n}\right)\,.
\end{aligned}$$

In general, this formula is also *incorrect*—one can check it disagrees with the $d = 1$ case of Corollary 3.18 except when $r = 1$, in which case both formulas agree with (2.9) since $G(e,e,n)$ is a duality group.

4. Conjectured explicit basis

Here we strengthen Theorem 1.1 to conjecture an explicit basis of the space of invariants $(S \otimes \wedge V^* \otimes \wedge V)^W$, extending ideas of [33]. In fact, for all of the remaining coincidental groups W (not type A nor $G(d,1,n)$), we verify this stronger conjecture in later sections to complete the verification of Theorem 1.1. Let us view

$$M_r := (S(V^*) \otimes \wedge V^* \otimes \wedge^r V)^W$$

as a module over the exterior algebra $(S(V^*)\otimes\wedge V^*)^W=\wedge_{S(V^*)^W}\{df_1,\dots,df_n\}$ via multiplication into the first two tensor factors. We require notation for letting a derivation act as an operator on M_r by taking partial derivatives of polynomial coefficients: Given any derivation $\theta=\sum_{j=1}^n h_j(x_1,\dots,x_n)\otimes y_j$ in $S(V^*)\otimes V$, let

$$\tilde\theta : S(V^*)\otimes\wedge^k V^*\otimes\wedge V \longrightarrow S(V^*)\otimes\wedge^{k+1}V^*\otimes\wedge V$$

be the differential operator defined by

$$f\otimes\eta\otimes\eta' \qquad\longmapsto\qquad \sum_{j=1}^n \overline{h}_j\left(\frac{\partial}{\partial x_1},\dots,\frac{\partial}{\partial x_n}\right)(f)\otimes x_j\wedge\eta\otimes\eta',$$

with bar indicating that complex coefficients are conjugated. Here, each x_i^m in the polynomial h_j is replaced by the iterated partial derivative $\partial^m/\partial x_i^m$ and the resulting operator is applied to f.

Note that in the case where θ is the Euler derivation $\theta_E=\sum_{j=1}^n x_i\otimes y_j$ in $S(V^*)\otimes V$, the operator $\tilde\theta_E$ restricts to the the usual exterior derivative $d: S(V^*)\mapsto S(V^*)\otimes V^*$. Restricting $\tilde\theta_E$ merely to a map $S(V^*)\otimes\wedge^0V^*\otimes V\to S\otimes\wedge^1V^*\otimes V$ gives the operator $\psi\mapsto d\psi$ from [28]. We can now state our conjecture.

Conjecture 4.1. *For W a coincidental reflection group, one may choose basic invarints $f_1,\dots,f_n$ and basic derivations $\theta_1,\dots,\theta_n$ so that each $M_r=(S(V^*)\otimes\wedge V^*\otimes\wedge^r V)^W$ for $r=1,\dots,n$ is a free module over the exterior algebra $R_r:=\wedge_{S(V^*)^W}\{df_1,\dots,df_{n-r}\}$ with basis*

$$\tilde\theta_{i_1}\cdots\tilde\theta_{i_m}\left(\theta_{j_1}\wedge\cdots\wedge\theta_{j_r}\right)$$

for $0\le m\le r$ and $1\le i_1<\cdots<i_m\le r$ and $1\le j_1<\cdots<j_r\le n$. For $r=0$, take basis element 1.

Equivalently one can write the conjectured basis more directly in terms of subsets of $\{1,\dots,n\}$. Define

$$\begin{aligned}
\tilde\theta_I &:= \tilde\theta_{i_1}\cdots\tilde\theta_{i_m} : S(V^*)\otimes\wedge^kV^*\otimes\wedge V\longrightarrow S(V^*)\otimes\wedge^{k+m}V^*\otimes\wedge V && \text{for } I=\{i_1<\cdots<i_m\},\\
\theta_J &:= \theta_{j_1}\wedge\cdots\wedge\theta_{j_r} \quad \text{in } S(V^*)\otimes\wedge^0\otimes\wedge^r V && \text{for } J=\{j_1<\cdots<j_r\},\\
df_L &:= df_{\ell_1}\wedge\cdots\wedge df_{\ell_{m'}} \ \text{in } S(V^*)\otimes\wedge^{m'}V^*\otimes\wedge^0 V && \text{for } L=\{\ell_1<\cdots<\ell_{m'}\}.
\end{aligned}$$

Conjecture 4.1 is then equivalent to the following statement.

Conjecture 4.1$'$. *For W a coincidental reflection group, one may choose basic invariants $f_1,\dots,f_n$ and basic derivations $\theta_1,\dots,\theta_n$ so that*

$$M_{r,k}:=(S(V^*)\otimes\wedge^kV^*\otimes\wedge^rV)^W \quad \textit{for} \quad 1\le r,k\le n \tag{4.2}$$

is a free module over $S(V^*)^W$ with basis

$$\{df_L\cdot\tilde\theta_I\,(\theta_J)\}$$

as one runs through all triples (I,J,L) of subsets $J\subset[n]$, $I\subset[r]$, $L\subset[n-r]$ with $|J|=r$ and $|I|+|L|=k$.

Remark 4.3. Note that several special cases of the equivalent Conjectures 4.1 and 4.1$'$ hold more generally:

- Conjecture 4.1 at $r=0$ holds for all reflection groups by [35].
- Conjecture 4.1 at $r=1$ holds for all duality groups by [28, Thm 1.1].
- Conjecture 4.1$'$ at $k=0$ holds for all reflection groups by [25, Thm. 3.1].

Proposition 4.4. *Conjecture 4.1$'$ (or its equivalent Conjecture 4.1) implies Theorem 1.1.*

Proof. The conjecture gives an S^W-basis for $(S\otimes\wedge V^*\otimes\wedge^rV)^W$ where we again appreviate $S=S(V^*)$; we compare the polynomial degrees of basis elements in the conjecture with the coefficient of s^r on the right in Theorem 1.1. Since $\mathrm{Hilb}(S^W,q)=\prod_{i=1}^n(1-q^{d_i})^{-1}$, after clearing the denominator in Theorem 1.1, we have

$$\left(\prod_{\ell=1}^{n-r}(1+q^{e_\ell}t)\right)\left(\prod_{i=1}^{r}(1+q^{-e_i^*}t)\right)\sigma_r(q^{e_1*},\dots,q^{e_n^*}). \tag{4.5}$$

We check that this matches the (q,t)-bidegrees of the S^W-basis elements $\{df_L \cdot \tilde{\theta}_I(\theta_J)\}$ in Conjecture 4.1′:

$$\left(\prod_{\ell=1}^{n-r}(1+q^{e_\ell}t)\right)\left(\prod_{i=1}^{r}(1+q^{-e_i^*}t)\right)\sigma_r(q^{e_1*},\dots,q^{e_n^*})$$

$$=\left(\sum_{L\subset[n-r]} t^{|L|}q^{\sum_{\ell\in L}e_\ell}\right)\left(\sum_{I\subset[r]} t^{|I|}q^{-\sum_{i\in I}e_i^*}\right)\left(\sum_{\substack{J\subset[n]:\\|J|=r}} q^{\sum_{j\in J}e_j^*}\right)=\sum_{(I,J,L)} t^{|I|+|L|}q^{\sum_{\ell\in L}e_\ell+\sum_{j\in J}e_j^*-\sum_{i\in I}e_i^*},$$

where the last sum runs through (I,J,L) satisfying the conditions in Conjecture 4.1′. Theorem 1.1 then follows since $df_L \cdot \tilde{\theta}_I(\theta_J)$ has $\wedge V^*$-degree $|I|+|L|$ and S-degree $\sum_{\ell\in L}e_\ell+\sum_{j\in J}e_j^*-\sum_{i\in I}e_i^*$. $\square$

5. The elements in Conjecture 4.1 are invariant

As a precursor to verifying Conjecture 4.1 or 4.1′, we check that the forms indicated there are indeed invariant under the action of any reflection group W.

Lemma 5.1. *For $V=\mathbb{C}^n$, let $W\subset GL(V)$ be a group of isometries and let $S=S(V^*)$.*

(a) The map $(S\otimes V\otimes 1)\times(S\otimes\wedge^k V^*\otimes\wedge^r V)\to S\otimes\wedge^{k+1}V^*\otimes\wedge^r V$, $(\theta,\omega)\mapsto\widetilde{\theta}(\omega)$, is W-equivariant.

(b) For θ in $(S\otimes 1\otimes V)^W$ and ω in $(S\otimes\wedge^k V^*\otimes\wedge^r V)^W$, the form $\tilde{\theta}(\omega)$ lies in $(S\otimes\wedge^{k+1}V^*\otimes\wedge^r V)^W$.

Proof. For any polynomials h, f in S, let $\partial(h)(f):=\bar{h}\big(\frac{\partial}{\partial x_1},\dots,\frac{\partial}{\partial x_n}\big)(f)$. The map $S\times S\to S$ given by $(h,f)\mapsto\partial(h)(f)$ is W-equivariant (see [33, §6]): For g in W,

$$g\big(\partial(h)f\big)=\partial(gh)(gf)\,.$$

Now consider some $\theta=h\otimes 1\otimes y_j$ in $S\otimes 1\otimes V$ and $\omega=f\otimes\eta\otimes\eta'$ in $S\otimes\wedge V^*\otimes\wedge V$. For any g in W, $(g\theta,g\omega)$ maps to $g(\tilde{\theta}(\omega))$ under the function in (a) since

$$\begin{aligned}g\big(\tilde{\theta}(\omega)\big)&=g\big(\partial(h)f\otimes x_j\wedge\eta\otimes\eta'\big)=g\big(\partial(h)f\big)\otimes gx_j\wedge g\eta\otimes g\eta'\\&=\partial\big(g(h)\big)(gf)\otimes g(x_j\wedge\eta)\otimes g\eta'=\widetilde{g\theta}\big(g(f\otimes\eta\otimes\eta')\big)=\widetilde{g\theta}(g\omega)\,.\end{aligned}$$

Thus $(\theta,g\omega)$ maps to $g\big(\tilde{\theta}(\omega)\big)$ whenever θ is W-invariant, and $\tilde{\theta}(\omega)$ is W-invariant whenever θ and ω are. $\square$

Remark 5.2. Recall that for any linear transformation g in $\mathrm{GL}(V)$, the matrix recording the dual action of g on V^* with respect to a dual basis is just the inverse transpose of the matrix recording the action of g on V. As W is finite, we may and have assumed the matrices giving the action of W are all unitary, so that the matrix of g in W acting on V^* is the complex conjugate (entry-wise) of the matrix of g acting on V. This explains why we take the complex conjugates of coefficients in defining the operators $\tilde{\theta}$.

6. Conjecture 4.1 agrees with the Gutkin-Opdam calculation

As a second precursor to verifying Conjecture 4.1 (or 4.1′), we check that it correctly predicts the sum of the degrees of the homogeneous elements in a basis for $(S\otimes\wedge^k V^*\otimes\wedge^r V)^W$ over $S(V^*)^G$ when W is a duality group. For any reflection group W, set (e.g., see [5, §4.5.5, Remark 4.48])

$$\begin{aligned}N&:=\#\{\text{reflections in }W\}=\textstyle\sum_{i=1}^n e_i,\\N^*&:=\#\{\text{reflecting hyperplanes for }W\}=\textstyle\sum_{i=1}^n e_i^*\,.\end{aligned}\tag{6.1}$$

Recall that for any W-representation U, the sum of the U-exponents

$$\psi(U)=e_1(U)+\ldots+e_{\dim U}(U)$$

is the sum of the polynomial degrees of homogeneous elements in any basis of $(S(V^*)\otimes U)^W$ as a free module over $S(V^*)^W$. The Gutkin-Opdam Lemma [5, Prop. 4.3.3, eqn. (4.6)] allows one to predict $\psi(U)$ as follows:

$$\begin{aligned}\psi(U)&=\sum_H\sum_{j=0}^{e_H-1} j\cdot\langle U\downarrow^W_{W_H},\det{}^j\rangle_{W_H}\\&=\sum_H D_H(U\downarrow^W_{W_H}),\end{aligned}$$

where H runs through the reflecting hyperplanes for W, with $W_H \subset W$ the pointwise stabilizer subgroup of H and $e_H := |W_H|$. Here D_H is the linear functional on the Grothendieck group $G_0(W_H)$ of W_H-representations that sends the 1-dimensional representation $\det^j$ to j for $j = 0, 1, \ldots, e_H - 1$.

Lemma 6.2. *Let W be a reflection group on V and $U = \wedge^k V^* \otimes \wedge^r V$. Then*

$$\psi(U) = \binom{n-1}{k-1}\binom{n-1}{r} N + \binom{n-1}{k}\binom{n-1}{r-1} N^* .$$

Proof. Let H be a reflecting hyperplane of W. The restrictions of V^* or V to W_H each contain $n-1$ copies of the trivial W_H-representation as direct summands, and then one extra summand carrying the 1-dimensional representations $\det^{e_H-1}$ or det, respectively. We sum over all k, r, apply the Gutkin-Opdam Lemma, and then extend D_H to be linear over $\mathbb{C}[s,t]$ keeping (6.1) in mind:

$$\begin{aligned}
\sum_{k,r} \psi(\wedge^k V^* \otimes \wedge^r V)\ t^k s^r &= \sum_H \sum_{k,r} D_H(\wedge^k V^* \otimes \wedge^r V \downarrow^W_{W_H})\ t^k s^r \\
&= \sum_H D_H\left((1+t)^{n-1}(1+t\det{}^{e_H-1})\,(1+s)^{n-1}(1+s\det)\right) \\
&= (1+t)^{n-1}(1+s)^{n-1}\sum_H D_H\left((1+t\det{}^{e_H-1})(1+s\det)\right) \\
&= (1+t)^{n-1}(1+s)^{n-1}\sum_H D_H\left((1+t\det{}^{e_H-1}+s\det+st)\right) \\
&= (1+t)^{n-1}(1+s)^{n-1}\sum_H (t(e_H-1)+s) \\
&= (1+t)^{n-1}(1+s)^{n-1}(tN+sN^*) .
\end{aligned}$$

The result then follows from extracting the coefficient of $t^k s^r$. ☐

Proposition 6.3. *Either of Theorem 1.1 or Conjecture 4.1 correctly predicts $\psi(U)$ for each $U = \wedge^k V^* \otimes \wedge^r V$ with $0 \le k, r \le n$ for any irreducible duality reflection group W.*

Proof of Proposition 6.3. On one hand, $\sum_{k,r} \psi(U) t^k s^r = (1+t)^{n-1}(1+s)^{n-1}(tN+sN^*)$ by Lemma 6.2. We fix $r = 0, 1, \ldots, n$ and extract the coefficient of s^r to obtain

$$\sum_k \psi(\wedge^k V^* \otimes \wedge^r V)\ t^k = (1+t)^{n-1}\left(tN\binom{n-1}{r} + N^*\binom{n-1}{r-1}\right).$$

On the other hand, Theorem 1.1 predicts for each $r = 0, \ldots, n$ that

$$\begin{aligned}
\sum_k \psi(\wedge^k V^* \otimes \wedge^r V)\ t^k &= \lim_{q\to 1}\left[\tfrac{\partial}{\partial q}\left(\sigma_r(q^{e_1^*}, \ldots, q^{e_n^*}) \cdot \prod_{i=1}^{n-r}(1+q^{e_i}t)\cdot \prod_{i=1}^{r}(1+q^{-e_i^*}t)\right)\right] \\
&= \sum_{\substack{J\subset\{1,2,\ldots,n\}:\\ |J|=r}} \lim_{q\to 1}\left[\tfrac{\partial}{\partial q}\left(q^{\sum_{j\in J} e_j^*}\cdot \prod_{i=1}^{n-r}(1+q^{e_i}t)\cdot\prod_{i=1}^{r}(1+q^{-e_i^*}t)\right)\right] \\
&= \sum_{\substack{J\subset\{1,2,\ldots,n\}:\\ |J|=r}} \left[(1+t)^n\sum_{j\in J} e_j^* \quad + \quad (1+t)^{n-1}\sum_{i=1}^{n-r} te_i \quad + \quad (1+t)^{n-1}\sum_{i=1}^{r}(-te_i^*)\right] \\
&= (1+t)^{n-1}\left[\sum_{i=1}^{n} e_i^*(1+t)\binom{n-1}{r-1} + t\binom{n}{r}\left(\sum_{i=1}^{n-r} e_i - \sum_{i=1}^{r} e_i^*\right)\right] .
\end{aligned}$$

It only remains to check the bracketed expression is $tN\binom{n-1}{r} + N^*\binom{n-1}{r-1}$. We use (6.1):

$$\begin{aligned}
\sum_{i=1}^{n} e_i^*(1+t)\binom{n-1}{r-1} &+ t\binom{n}{r}\left(\sum_{i=1}^{n-r} e_i - \sum_{i=1}^{r} e_i^*\right) \\
&= N^*(1+t)\binom{n-1}{r-1} + t\binom{n}{r}(N-rh) \\
&= t\left(\binom{n-1}{r-1}N^* + \binom{n}{r}(N-rh)\right) + N^*\binom{n-1}{r-1} \\
&= t\left(\binom{n-1}{r-1}N^* + \binom{n-1}{r-1}(N-rh) + \binom{n-1}{r}(N-rh)\right) + N^*\binom{n-1}{r-1} \\
&= tN\binom{n-1}{r} + N^*\binom{n-1}{r-1}.
\end{aligned}$$

Here we used the fact that $N + N^* = nh$ (as W is a duality group) and $\binom{n-1}{r-1}(n-r) = \binom{n-1}{r}r$. $\square$

Independence over the fraction field. We now explain why the above Opdam-Gutkin calculation implies that Conjecture 4.1 may be shown with an independence argument.

Lemma 6.4. [28, Lemma 4.1] *Let A be a graded k-algebra and integral domain, and $M \cong A^p$ a free graded A-module whose homogeneous basis elements have degrees suming to ψ. Then another set of homogeneous elements $\{n_1, \cdots, n_p\}$ in M with the same degree sum $\sum_{i=1}^{p}\deg(n_i) = \psi$ form an A-basis for M if and only if they are linearly independent over the fraction field $K = Frac(A)$.*

Thus one may verify Conjecture 4.1 by proving that the basis elements of the module $(S(V^*) \otimes \wedge^k V^* \otimes \wedge^r V)^W$ it predicts are linearly independent over the fraction field of $(S(V^*)$ for each k, r:

Proposition 6.5. *Let W be a coincidental reflection group with any set of basic invariants $f_1, \dots, f_n$ and basic derivations $\theta_1, \dots, \theta_n$. Fix some $1 \le r \le n$. If the derviation differential forms*

$$\tilde{\theta}_{i_1} \cdots \tilde{\theta}_{i_k}\left(\theta_{j_1} \wedge \cdots \wedge \theta_{j_r}\right) \quad \in S(V^*) \otimes \wedge^k V^* \otimes \wedge^r V$$

for $0 \le k \le r$ and $1 \le i_1 < \cdots < i_k \le r$ and $1 \le j_1 < \cdots < j_r \le n$ are linearly independent over the fraction field $Frac(S(V^*))$, then they form a basis for $M_r = (S(V^*) \otimes \wedge V^* \otimes \wedge^r V)^W$ over the exterior subalgebra $R_r := \wedge_{S(V^*)^W}\{df_1, \dots, df_{n-r}\}$. In this case, M_r is a free module over R_r.

Proof. Use Proposition 6.3 together with Lemma 6.4 and Lemma 5.1. $\square$

By Proposition 6.5, it suffices to check various determinants are nonzero to prove Conjecture 4.1.

7. The main result and Molchanov's hypothesis

In this section, we outline the proof of our main result, Theorem 1.1, from the Introduction:

Theorem 1.1. For any coincidental complex reflection group W acting on $V = \mathbb{C}^n$,

$$\mathrm{Hilb}\left((S(V^*) \otimes \wedge V^* \otimes \wedge V)^W, q, t, s\right) = \sum_{r=0}^{n} \sigma_r(q^{e_1^*}, \dots, q^{e_n^*}) \frac{\prod_{i=1}^{r}(1+q^{e_i^*}t)\ \prod_{i=1}^{n-r}(1+q^{e_i}t)s^r}{\prod_{i=1}^{n}(1-q^{d_i})}.$$

Proof. We proceed in essentially three cases, some of which prove the stronger Conjecture 4.1′; see Lemma 4.4.

- For the Weyl groups of type A_n and the Shephard-Todd family $G(d,1,n)$, Theorem 1.1 was deduced in Section 3, as Corollaries 3.4 and 3.16.
- We prove Conjecture 4.1′ uniformly in Section 8 for all coincidental groups of rank 2, relying on Proposition 6.5 so as to only check that certain determinants are nonzero; see Theorem 8.4.
- This leaves the exceptional real type H_3 and Shephard groups G_{25}, G_{26}, G_{32} of ranks 3 or 4, where we checked Conjecture 4.1′ in `Mathematica` via Proposition 6.5, for these choices of $\{f_i\}, \{\theta_i\}$:
  - For H_3, we used $\{f_i\}$ from Saito, Yano, and Sekiguchi [30], with $\theta_i = \sum_{j=1}^{n} \frac{\partial f_i}{\partial x_j} \otimes y_j$.
  - For G_{25}, G_{26}, G_{32}, we used $\{f_i\}$ and $\{\theta_i\}$ from Orlik and Terao [26, Appendix B.3]. $\square$

Remark 7.1. Theorem 1.1 is very closely related to a hypothesis stated by Molchanov [24, §8]. Molchanov's formulation[1] differs from Theorem 1.1 both in that he assumes that W is a *real* reflection group (so that as W-representations, $V^* \cong V$ and $e_i^* = e_i$) and in that he assumes that the exponents are *distinct*. Unfortunately, the scope of his hypothesis is off, and Theorem 1.1 seems to be the correct formulation. In fact, combining Remark 3.19 with the data on the noncoincidental exceptional reflection groups presented in Section 11, one sees that the formula in Theorem 1.1 holds for an irreducible reflection group W, real or complex, *if and only W is coincidental.*

8. Reflection groups in two dimensions

Here we verify Conjecture 4.1′ for rank 2 coincidental reflection groups using Proposition 6.5. In two dimensions, the coincidental groups W are the same as the irreducible duality groups. Recall as notaion that W acts on $V = \mathbb{C}^2$ with $\mathbb{C}$-basis $\{y_1, y_2\}$ and on V^* with dual $\mathbb{C}$-basis $\{x_1, x_2\}$, so that $S(V^*) = \mathbb{C}[x_1, x_2]$.

Lemma 8.1. *For W any rank 2 irreducible finite reflection group acting on $V = \mathbb{C}^2$, the W-invariant space $\left(V^* \otimes V^* \otimes \wedge^2 V\right)^W$ is 1-dimensional over $\mathbb{C}$, spanned by*

$$(x_1 \otimes x_2 - x_2 \otimes x_1) \otimes y_1 \wedge y_2. \tag{8.2}$$

Proof. Again, we write $S = S(V^*)$. As a $\mathrm{GL}(V)$-representation, $V^* \otimes V^* \cong S^2 \oplus \wedge^2 V^*$. This isomorphism restricts one of W-representations, and hence

$$\left(V^* \otimes V^* \otimes \wedge^2 V\right)^W \cong \left(S^2 \otimes \wedge^2 V\right)^W \oplus \left(\wedge^2 V^* \otimes \wedge^2 V\right)^W .$$

We analyze the two direct summands in this last expression. Since $\wedge^2 V$ is 1-dimensional spanned by $dy_1 \wedge dy_2$ and carries the W-character det, for any W-representation U, the W-fixed space $\left(U \otimes \wedge^2 V\right)^W$ will be the tensor product of the $\det^{-1}$-isotypic component of U with $\mathbb{C} dy_1 \wedge dy_2$.

For $U = \wedge^2 V^*$, this $\det^{-1}$-isotypic component is 1-dimensional, spanned by $x_1 \otimes x_2 - x_2 \otimes x_1$.

For $U = S^2$, we argue that this $\det^{-1}$-isotypic component will vanish. For reflection groups W of any rank, the $\det^{-1}$-isotypic component of S is the free S^W-module $S^W \cdot Q$ of rank 1 (by Stanley [38, Thm. 3.1], see also [26, Ex. 6.40]) for Q the product of the linear forms defining the reflecting hyperplanes with degree $N^* = e_1^* + \cdots + e_n^*$, the number of reflecting hyperplanes. But $N^* > 2$ for any irreducible reflection group W of rank at least 2 and hence S^2 has zero intersection with this $\det^{-1}$-isotypic component.

Thus $\left(V^* \otimes V^* \otimes \wedge^2 V\right)^W \cong \left(\wedge^2 V^* \otimes \wedge^2 V\right)^W$ is spanned by $(x_1 \otimes x_2 - x_2 \otimes x_1) \otimes y_1 \wedge y_2$. ☐

Proposition 8.3. *If $h(x_1, \ldots, x_n)$ in $S(V^*)$ is homogeneous and nonzero, then $\bar{h}(\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n})(h) \neq 0$.*

Proof. Express h of degree d as a finite sum of monomials,

$$h = \sum_{\substack{\alpha=(\alpha_1,\ldots,\alpha_n):\\ \sum_i \alpha_i = d}} c_\alpha x_1^{\alpha_1} \cdots x_n^{\alpha_n} = \sum_\alpha c_\alpha \mathbf{x}^\alpha,$$

and write $(\partial x_i)^m$ for $\partial^m / \partial x_i^m$. For $\sum_i \alpha_i = \sum_i \beta_i$, the monomial $(\partial^{\alpha_1})^{\alpha_1} \cdots (\partial x_n)^{\alpha_n} (\mathbf{x}^\beta)$ is $\alpha_1! \cdots \alpha_n!$ if $\beta = \alpha$ but vanishes for $\beta \neq \alpha$. Thus by linearity,

$$\bar{h}(\partial x_1, \ldots, \partial x_n)(h) = \sum_{\alpha,\beta} \bar{c}_\alpha \, c_\alpha \; (\partial x_1)^{\alpha_1} \cdots (\partial x_n)^{\alpha_n} (\mathbf{x}^\beta) = \sum_\alpha |c|^2 \; \alpha_1! \cdots \alpha_n!\,,$$

which is strictly positive as long as at least one $c_\alpha \neq 0$ in $\mathbb{C}$, that is, as long as $h \neq 0$ in S. ☐

Proposition 8.4. *Conjecture 4.1 holds for any irreducible duality group W acting on $\mathbb{C}^2$: For $S = S(V^*)$ and $M_r = (S \otimes \wedge V^* \otimes \wedge^r V)^W$, there is a set of basic invariants f_1, f_2 and basic derivations θ_1, θ_2 so that*

- *M_0 is free over $R_0 = \wedge_{S^W}\{df_1, df_2\}$ with basis $\{1\}$;*
- *M_1 is free over $R_1 = S^W df_1$ with basis $\{\theta_1,\, \theta_2,\, \tilde{\theta}_1(\theta_1),\, \tilde{\theta}_1(\theta_2)\}$;*
- *M_2 is free over $R_2 = S^W$ with basis $\{\theta_1 \wedge \theta_2,\, \tilde{\theta}_1(\theta_1 \wedge \theta_2),\, \tilde{\theta}_2(\theta_1 \wedge \theta_2),\, \tilde{\theta}_1\tilde{\theta}_2(\theta_1 \wedge \theta_2)\}$.*

[1]Warning: translation of his paper to English introduced two typos—instead of $\prod_{i=1}^r$ and $\prod_{i=1}^n$, it has $\prod_{i=1}^m$ for both.

Proof. Recall from Remark 4.3 that Conjecture 4.1 is known for all duality groups when $r \in \{0,1\}$ or $k=0$. Since $0 \le r \le n = 2$, it hence suffices to only consider here the case $r = n = 2$ and $k \in \{1,2\}$.

By Proposition 6.5, we need only check the linear independence over the fraction field K of S of

$$\tilde{\theta}_1(\theta_1 \wedge \theta_2) \text{ and } \tilde{\theta}_2(\theta_1 \wedge \theta_2) \quad \text{in} \quad M_{2,1} := (S \otimes \wedge^1 V^* \otimes \wedge^2 V)^W \quad \text{(the case } k=1)$$

and also check that

$$0 \neq \tilde{\theta}_1\tilde{\theta}_2(\theta_1 \wedge \theta_2) \quad \text{in} \quad M_{2,2} := (S \otimes \wedge^2 V^* \otimes \wedge^2 V)^W \quad \text{(the case } k=2)\,.$$

Recall that $\theta_1 \wedge \theta_2 = Q \otimes 1 \otimes y_1 \wedge y_2$ by [25, Thm. 3.1] as $\wedge^2 V = \mathbb{C} y_1 \wedge y_2$. Again, we write ∂x_i for $\partial/\partial x_i$.

The case $n = r = 2$ and $k = 2$. The derivation differential form

$$\tilde{\theta}_1\tilde{\theta}_2(\theta_1 \wedge \theta_2) = \tilde{\theta}_1\tilde{\theta}_2(Q \otimes 1 \otimes y_1 \wedge y_2) = \bar{Q}(\partial x_1, \partial x_2)(Q) \otimes x_1 \wedge x_2 \otimes y_1 \wedge y_2$$

is nonzero since Lemma 8.3 implies that the scalar $\bar{Q}(\partial x_1, \partial x_2)(Q)$ in $\mathbb{C}$ is nonzero.

The case $n = r = 2$ and $k = 1$. Here, one must check that

$$\begin{aligned} \omega_1 &:= \tilde{\theta}_1(\theta_1 \wedge \theta_2) = \tilde{\theta}_1(Q \otimes 1 \otimes y_1 \wedge y_2) \quad \text{and} \\ \omega_2 &:= \tilde{\theta}_2(\theta_1 \wedge \theta_2) = \tilde{\theta}_2(Q \otimes 1 \otimes y_1 \wedge y_2) \end{aligned} \tag{8.5}$$

are K-linearly independent in $S \otimes V^* \otimes \wedge^2 V$. Our strategy is to make convenient choices for basic derivations θ_1, θ_2 that identify ω_1, ω_2 more concretely. As explained in [26, Appendix B.2], one may choose in rank 2

$$\begin{aligned} \theta_1 &= x_1 \otimes 1 \otimes y_1 + x_2 \otimes 1 \otimes y_2 \quad \text{and} \\ \theta_2 &= -\tfrac{\partial Q}{\partial x_2} \otimes 1 \otimes y_1 + \tfrac{\partial Q}{\partial x_1} \otimes 1 \otimes y_2\,. \end{aligned} \tag{8.6}$$

With this choice,

$$\begin{aligned} \omega_1 &= \left(\tfrac{\partial Q}{\partial x_1} \otimes x_1 + \tfrac{\partial Q}{\partial x_2} \otimes x_2\right) \otimes y_1 \wedge y_2 \quad \text{and} \\ \omega_2 &= \left(-\overline{\tfrac{\partial Q}{\partial x_2}}(\partial x_1, \partial x_2)(Q) \otimes x_1 + \overline{\tfrac{\partial Q}{\partial x_1}}(\partial x_1, \partial x_2)(Q) \otimes x_2\right) \otimes y_1 \wedge y_2. \end{aligned} \tag{8.7}$$

Note that ω_2 has S-degree 1 inside $S \otimes V^* \otimes \wedge^2 V^*$. Also notice that $\omega_2 \neq 0$, since otherwise

$$\overline{\tfrac{\partial Q}{\partial x_1}}(\partial x_1, \partial x_2)(Q) = \overline{\tfrac{\partial Q}{\partial x_2}}(\partial x_1, \partial x_2)(Q) = 0$$

which would then imply the following contradiction to Proposition 8.3:

$$\begin{aligned} 0 = \tfrac{\partial}{\partial x_1}\left(\overline{\tfrac{\partial Q}{\partial x_1}}(\partial x_1, \partial x_2)(Q)\right) + \tfrac{\partial}{\partial x_2}\left(\overline{\tfrac{\partial Q}{\partial x_2}}(\partial x_1, \partial x_2)(Q)\right) &= \left(\overline{x_1\tfrac{\partial Q}{\partial x_1}} + \overline{x_2\tfrac{\partial Q}{\partial x_2}}\right)(\partial x_1, \partial x_2)(Q) \\ &= \deg(Q) \cdot \bar{Q}(\partial x_1, \partial x_2)(Q)\,. \end{aligned}$$

Thus ω_2 is a nonzero element of $\left(V^* \otimes V^* \otimes \wedge^2 V\right)^W$ by Lemma 5.1 and we may identify ω_2, up to a nonzero scalar, with $(x_1 \otimes x_2 - x_2 \otimes x_1) \otimes y_1 \wedge y_2$ by Lemma 8.1.

Finally, we check that ω_1, ω_2 are K-linearly independent. The matrix expressing ω_1, ω_2 with respect to the S-basis $x_1 \otimes y_1 \wedge y_2$, $x_2 \otimes y_1 \wedge y_2$ of $S \otimes V^* \otimes \wedge^2 V$ has determinant $x_1\frac{\partial Q}{\partial x_1} + x_2\frac{\partial Q}{\partial x_2} = \deg(Q) \cdot Q \neq 0$. ☐

9. Conversion between (q,t)-analogues of h-vectors and f-vectors

We now highlight some applications of Theorem 1.1 to f-vectors and h-vectors in algebraic combinatorics. We will see in Remark 10.6 later how these applications answer a question on q-Kirkman and q-Narayana numbers raised in [2, Problem 11.3].

Let W be a coincidental reflection group with smallest exponent e_1 and exponent gap a. In the Introduction, we defined these (q,t)-analogues of the f-vector and h-vector:

$$\begin{aligned} f_r(W;q,t) &= q^{r+a\binom{r}{2}} \begin{bmatrix} n \\ r \end{bmatrix}_{q^a} \frac{(-tq^{e_1};q^a)_{n-r}(-tq^{-1};q^{-a})_r}{(q^{e_1+1};q^a)_n} \quad &\big(\text{the } (q,t)\text{-analogue of the } f\text{-vector}\big), \\ h_r(W;q,t) &= (-tq^{-ar-1})^{n-r} \begin{bmatrix} n \\ r \end{bmatrix}_{q^a} \frac{(-tq^{-1};q^{-a})_r}{(q^{e_1+1};q^a)_r} \quad &\big(\text{the } (q,t)\text{-analogue of the } h\text{-vector}\big)\,. \end{aligned} \tag{9.1}$$

We later explain in Section 10 why it is appropriate to call these analogues of f-vectors and h-vectors (see (10.7) and (10.8)). In this section, we prove Theorem 1.5 of the Introduction that converts between these (q,t)-analogues. As preparation, we rephrase $f_r(W;q,t)$ using the notation of *basic hypergeometric functions* [9, Chap. 1]:

$$ {}_{k+1}\phi_k \left[\begin{matrix} a_1 & a_2 & \dots & a_{k+1} \\ & b_1 & \dots & b_k \end{matrix} \;\middle|\; q; z \right] := \sum_{r \geq 0} z^r \, \frac{(a_1;q)_r (a_2;q)_r \cdots (a_{k+1};q)_r}{(q;q)_r (b_1;q)_r \cdots (b_k;q)_r} \,. \tag{9.2} $$

Proposition 9.3. *The following two statements are equivalent reformulations of Theorem 1.1:*

$$ (a) \quad f_r(W;q,t) \;=\; \tfrac{1}{|W|} \sum_{w \in W} \frac{\det(1+tw)}{\det(1-qw)} \;=\; t^r \begin{bmatrix} n \\ r \end{bmatrix}_{q^a} \frac{(-tq^{e_1};q^a)_{n-r} (-qt^{-1};q^a)_r}{(q^{e_1+1};q^a)_n} \,. $$

$$ \begin{aligned} (b) \quad \sum_{r=0}^{n} s^r f_r(W;q,t) \;&=\; \tfrac{1}{|W|} \sum_{w \in W} \frac{\det(1+sw^{-1}) \det(1+tw)}{\det(1-qw)} \\ &=\; \frac{(-tq^{e_1};q^a)_n}{(q^{e_1+1};q^a)_n} \quad {}_2\phi_1 \left[\begin{matrix} q^{-an} & -qt^{-1} \\ & -q^{a(1-n)-e_1} t^{-1} \end{matrix} \;\middle|\; q^a ; -sq^{a-e_1} \right] . \end{aligned} $$

Proof. For Equation (a), we simply rewrite Theorem 1.1$'$ with

$$ f_r(W;q,t) = q^{r+a\binom{r}{2}} \begin{bmatrix} n \\ r \end{bmatrix}_{q^a} \frac{(-tq^{e_1};q^a)_{n-r} (-tq^{-1};q^{-a})_r}{(q^{e_1+1};q^a)_n} = t^r \begin{bmatrix} n \\ r \end{bmatrix}_{q^a} \frac{(-tq^{e_1};q^a)_{n-r} (-qt^{-1};q^a)_r}{(q^{e_1+1};q^a)_n} $$

by applying this easy fact (see [9, eqn. (I.3)]) to the numerator factor $(-tq^{-1};q^{-a})_r$:

$$ (z;q^{-1})_r = z^r \; q^{-\binom{r}{2}} \; (-z^{-1};q)_r \,. \tag{9.4} $$

To see (b), we reexpress (a) as

$$ f_r(W;q,t) = t^r \, \frac{(q^a;q^a)_n}{(q^{e_1+1};q^a)_n} \, \frac{(-tq^{e_1};q^a)_{n-r}}{(q^a;q^a)_{n-r}} \, \frac{(-qt^{-1};q^a)_r}{(q^a;q^a)_r} \,. $$

We rewrite the middle quotient in the product by applying the identity (see [9, (I.11)])

$$ \frac{(w;q)_{n-r}}{(z;q)_{n-r}} = \frac{(w;q)_n}{(z;q)_n} \, \frac{(q^{1-n}/z;q)_r}{(q^{1-n}/w;q)_r} \, \left(\frac{z}{w} \right)^r $$

with $w = -tq^{e_1}$, $z = q^a$ and obtain

$$ f_r(W;q,t) = \frac{(-tq^{e_1};q^a)_n}{(q^{e_1+1};q^a)_n} \, (-q^{a-e_1})^r \, \frac{(q^{-an};q^a)_r}{(q^a;q^a)_r} \, \frac{(-qt^{-1};q^a)_r}{(-q^{a(1-n)-e_1}t^{-1};q^a)_r} \,, $$

which then immediately gives (b) from definition (9.2) of ${}_2\phi_1$. ☐

We can now extend the relation between f- and h-vectors for n-dimensional simple polytopes, namely,

$$ \sum_{r=0}^{n} s^r f_r \;=\; \sum_{r=0}^{n} (1+s)^r \; h_r \,, \tag{9.5} $$

to a relation between the above (q,t)-analogues; we prove the theorem from the Introduction:

Theorem 1.5. *For any coincidental reflection group W with exponent gap a,*

$$ \sum_{r=0}^{n} s^r f_r(W;q,t) \;=\; \sum_{r=0}^{n} (-sq;q^a)_r \cdot h_r(W;q,t) \,. $$

Proof. We apply a terminating form of *Jackson's ${}_2\phi_1$-transformation*[2] [9, III.7],

$$ {}_2\phi_1 \left[\begin{matrix} q^{-n} & b \\ & c \end{matrix} \;\middle|\; q, z \right] \;=\; \frac{(c/b;q)_n}{(c;q)_n} \; {}_3\phi_2 \left[\begin{matrix} q^{-n} & b & bzq^{-n}/c \\ & bq^{1-n}/c & 0 \end{matrix} \;\middle|\; q, q \right] , \tag{9.6} $$

[2]The authors thank Dennis Stanton for pointing them to this identity.

to the ${}_2\phi_1$-expression in Proposition 9.3(b) after first replacing q by q^a and then selecting parameter choices

$$b = -qt^{-1}, \quad c = -q^{a(1-n)-e_1}t^{-1}, \quad z = -sq^{a-e_1}.$$

Since

$$c/b = q^{a(1-n)-e_1-1}, \quad bq^{1-n}/c = q^{e_1+1}, \quad bzq^{-1}/c = -sq,$$

and $(z;q)_n = (q^{1-n}/z;q)_n(-z)^n q^{\binom{n}{2}}$ (see [9, eqn. (I.7)]), one can rewrite the quotient in (9.6) as

$$\frac{(c/b;q)_n}{(c;q)_n} = \frac{(q^{a(1-n)-e_1-1};a^a)_n}{(-t^{-1}q^{a(1-n)-e_1};q^a)_n} = \frac{(q^{e_1+1};q^a)_n}{(-tq^{e_1};q^a)_n}(-tq^{-1})^n.$$

Thus Proposition 9.3(b) and (9.6) imply that

$$\begin{aligned}\sum_{r=0}^{n} s^r f_r(W;q,t) &= (-tq^{-1})^n \cdot {}_3\phi_2\left[\begin{matrix} q^{-an} & -qt^{-1} & -sq \\ & q^{e_1+1} & 0\end{matrix}\;\middle|\; q^a, q^a\right] \\ &= (-tq^{-1})^n \sum_{r=0}^{n}(-sq;q^a)_r \cdot q^{ar}\,\frac{(q^{-an};q^a)_r\,(-qt^{-1};q^a)_r}{(q^a;q^a)_r\,(q^{e_1+1};q^a)_r}.\end{aligned}$$

It remains to check that

$$(9.7) \qquad (-tq^{-ar-1})^{n-r}\begin{bmatrix} n \\ r\end{bmatrix}_{q^a}\frac{(-tq^{-1};q^{-a})_r}{(q^{e_1+1};q^a)_r} = (-tq^{-1})^n\, q^{ar}\,\frac{(q^{-an};q^a)_r\,(-qt^{-1};q^a)_r}{(q^a;q^a)_r\,(q^{e_1+1};q^a)_r}.$$

We substitute definition (1.2) of the q-binomial and cancel common factors to see that (9.7) is equivalent to

$$(-tq^{-ar-1})^{n-r}\,\frac{(q^a;q^a)_n}{(q^a;q^a)_{n-r}}\,(-tq^{-1};q^{-a})_r = (-tq^{-1})^n\, q^{ar}\,(q^{-an};q^a)_r\,(-qt^{-1};q^a)_r.$$

We verify this last equality by applying (9.4) to the factor $(-tq^{-1};q^{-a})_r$ on the left and rewriting the factor $(q^{-an};q^a)_r$ on the right using this fact (from [9, I.12]) with q replaced by q^a:

$$(9.8) \qquad (q^{-n};q)_r = \frac{(q;q)_n}{(q;q)_{n-r}}\,(-1)^r\, q^{\binom{r}{2}-nr}. \qquad \square$$

10. The connection with Catalan, Kirkman, Narayana, Cambrian and clusters

We explain here how specializations of our product formulas for $f_r(W;q,t)$ and $h_r(W;q,t)$ give known product formulas for q-Catalan, q-Kirkman, and q-Narayana numbers. We also see how their $q = 1$ specializations give the f-vectors and h-vectors for Cambrian and cluster fans. This starts with certain graded representations of a reflection group W called *graded parking spaces.*

The graded parking spaces. For a positive integer p, define a class function $\chi^{(p)} : W \longrightarrow \mathbb{Q}(q)$ by

$$\chi^{(p)}(w) := \frac{\det(1-q^p w)}{\det(1-qw)}.$$

For special values of p, the function $\chi^{(p)}$ is actually a $\mathbb{Z}[q]$-valued class function and even turns out to be the graded character of a genuine W-representation. Ito and Okada [17] tabulated the values of p for which this holds for each irreducible complex reflection group. For duality groups W, these special values of p include all the *Fuss-Catalan* cases, that is, cases where $p \equiv 1 \bmod h$ for $h = d_n = \max\{d_i\}$:

- This fact is related to work of Haiman [12, §7] for Weyl groups, where $\chi^{(p)}$ gives a graded version of the *p-parking space* W-permutation representation on R/pR, in which R is the root lattice for W.
- It holds more generally for real reflection groups W via results from representation theory of *rational Cherednik algebras*, e.g., see [3, Remark 4.4]; Gordon and Griffeth [10, §1.6] generalize these ideas to all complex reflection groups.
- One may verify this fact from Ito and Okada's tabulation [17, Table 1], where it only fails for the four *non-duality* groups G_{12}, G_{13}, G_{22}, and G_{31}.

The q-Kirkman numbers. In the cases where $\chi^{(p)}$ is a genuine graded character, setting $t=-q^p$ in Proposition 9.3(a) gives an expression for the (graded) multiplicity of the W-irreducibles $\wedge^r V$ in $\chi^{(p)}$:

$$f_r(W;q,-q^p) = \left\langle \chi^{(p)}, \wedge^r(V) \right\rangle_W = (-1)^r \; q^{pr} \; \begin{bmatrix} n \\ r \end{bmatrix}_{q^a} \frac{(q^{p+e_1};q^a)_{n-r}(q^{p-1};q^{-a})_r}{(q^{e_1+1};q^a)_n}. \tag{10.1}$$

For real reflection groups W, these graded multiplicities are called *q-Kirkman numbers*; see [2, §9, §11]. Specializing (10.1) to the case of types A (where $e_1=1=a$) and B/C (where $e_1=1, a=2$) gives

$$\begin{aligned} f_r(A_{n-1};q,-q^p) &= q^{\binom{r+1}{2}} \frac{1}{[p]_q} \begin{bmatrix} n-1 \\ r \end{bmatrix}_q \begin{bmatrix} p+n-r-1 \\ n \end{bmatrix}_q && \text{for } \gcd(n,p)=1 \text{ and} \\ f_r(B_n/C_n;q,-q^p) &= q^{r^2} \begin{bmatrix} \frac{p-1}{2} \\ r \end{bmatrix}_{q^2} \begin{bmatrix} \frac{p-1}{2}+n-r \\ n-r \end{bmatrix}_q && \text{for } p \text{ odd.} \end{aligned}$$

The special case $p=h+1$ was listed in [2]. In verifying that (10.1) specializes to these two formulas, it is helpful to note that $p \equiv 1 \bmod a$ in these cases, and so one can again use (9.8) to rewrite (10.1) as

$$f_r(W;q,-q^p) = q^{r+a\binom{r}{2}} \begin{bmatrix} n \\ r \end{bmatrix}_{q^a} \frac{(q^{p+e_1};q^a)_{n-r} \; (q^a;q^a)_{\frac{p-1}{a}}}{(q^{e_1+1};q^a)_n \; (q^a;q^a)_{\frac{p-1}{a}-r}}.$$

The q-Catalans and q-Narayanas. Setting $s=0$ in Theorem 1.5 gives the identity

$$\frac{(-tq^{e_1};q^a)}{(q^{e_1+1};q^a)_n} = f_0(W;q,t) = \sum_{r=0}^n h_r(W;q,t). \tag{10.2}$$

In the cases where $\chi^{(p)}$ is a genuine graded character, setting $t=-q^p$ in the left side of (10.2) gives what one might call the *p-rational version of the q-Catalan number* for W:

$$\frac{(q^{p+e_1};q^a)}{(q^{1+e_1};q^a)_n} = \prod_{i=1}^n \frac{1-q^{p+e_i}}{1-q^{1+e_i}} =: \mathrm{Cat}^{(p)}(W,q).$$

Here, the first equality assumes W is coincidental with smallest exponent e_1 and exponent gap q. When $p=h+1$, this product is called the *q-Catalan number for W*; see, e.g. Armstrong [1]. Thus setting $t=-q^p$ in (10.2) gives a summation formula for coincidental reflection groups W:

$$\mathrm{Cat}^{(p)}(W,q) = \frac{(q^{p+e_1};q^a)}{(q^{e_1+1};q^a)_n} = \sum_{r=0}^n h_r(W;q,-q^p),$$

where each summand has the explicit product formula

$$h_r(W;q,-q^p) = (q^{p-ar-1})^{n-r} \begin{bmatrix} n \\ r \end{bmatrix}_{q^a} \frac{(q^{p-1};q^{-a})_r}{(q^{e_1+1};q^a)_r}. \tag{10.3}$$

Happily, (10.3) agrees with the type A and B/C formulas for the *q-Narayana numbers* computed in [29]:

$$\begin{aligned} h_r(A_{n-1};q,-q^p) &= q^{(n-1-r)(p-1-r)} \frac{1}{[r+1]_q} \begin{bmatrix} n-1 \\ r \end{bmatrix}_q \begin{bmatrix} p-1 \\ r \end{bmatrix}_q && \text{for } \gcd(n,p)=1, \\ h_r(B_n/C_n;q,-q^p) &= (q^2)^{(n-r)(\frac{p-1}{2}-r)} \begin{bmatrix} n \\ r \end{bmatrix}_{q^2} \begin{bmatrix} \frac{p-1}{2} \\ r \end{bmatrix}_{q^2} && \text{for } p \text{ odd.} \end{aligned} \tag{10.4}$$

In verifying this, it is helpful to again use (9.4) to replace $(q^{p-1};q^{-a})_r$ in (10.3) with a multiple of $(q^{1-p};q^a)_r$ and then to further use that $p \equiv 1 \bmod a$ in these cases and employ (9.8) again, rewriting (10.3) as

$$h_r(W;q,-q^p) = q^{(n-r)(p-ar-1)} \begin{bmatrix} n \\ r \end{bmatrix}_{q^a} \frac{(q^a;q^a)_{\frac{p-1}{a}}}{(q^{e_1+1};q^a)_n \; (q^a;q^a)_{\frac{p-1}{a}-r}}.$$

Remark 10.5. We say "happily" above because the formulas in (10.4) came from a subtle and general *Weyl group* construction that arose from work of the third author [36], described in [29], that always produces q-Narayana numbers summing to q-Catalan numbers. But there was nothing, *a priori*, indicating that they must *coincide* with the values $h_r(W;q,-q^p)$ arising naturally here for each coincidental reflection group W.

Remark 10.6. For the coincidental types, setting $t = -q^p$ in Theorem 1.5 relates the q-Kirkman numbers $f_r(W; q, -q^p)$ to the q-Narayana numbers $h_r(W; q, -q^p)$, as asked for in [2, Problem 11.3].

The f-vector and h-vector. Further specializing to $t = -q^{h+1}$ (so $p = h+1$) and taking $q \to 1$ in (9.1) produces integers for a *real* reflection group W

$$f_r := \left[f_r(W; q, -q^{h+1})\right]_{q=1} \quad \text{and} \tag{10.7}$$

$$h_r := \left[h_r(W; q, -q^{h+1})\right]_{q=1} \tag{10.8}$$

that were observed in [2, §3.3] to be the *f-vector* and *h-vector*, respectively, for the (finite type) *cluster complexes* of Fomin and Zelevinsky (in the Weyl group case) and *Cambrian fans* of Reading (for arbitrary real reflection groups). Specifically, f_r counts the number of clusters of cardinality $n - r$, or the number of cones in the fan having dimension $n - r$. Thus Theorem 1.5 specializes in this instance to the usual h-vector-to-f-vector relationship for the simplicial spheres associated to these fans, or to the simple polytopes for which they are the normal fans, constructed by Hohlwedg, Lange, and Thomas [16]. This again answers the second part of the question raised in [2, Problem 11.3] for (real) coincidental groups W.

Remark 10.9. Theorem 1.1 explains a mysterious product formula observed by Fomin and Reading [7, Thm. 8.5 at $m = 1$] for the number of r-dimensional cones in the cluster/Cambrian fan for real coincidental reflection groups W:

$$f_r = \binom{n}{r} \prod_{i=1}^{n-r} \frac{h + d_i}{d_i}.$$

The formula follows from computing (10.7) by setting $t = -q^{h+1}$ and then $q = 1$ in Theorem 1.1:

$$f_r = \left[f_r(W; q, -q^{h+1})\right]_{q=1} = \binom{n}{r} \frac{\prod_{i=1}^{r}(h + 1 - e_i^*) \cdot \prod_{i=1}^{n-r}(h + 1 + e_i)}{\prod_{i=1}^{n} d_i} = \binom{n}{r} \prod_{i=1}^{n-r} \frac{h + d_i}{d_i}$$

where the last equality uses the fact that $d_i = e_i + 1$ and that $e_i^* = h - e_{n+1-i}$ for real reflection groups W.

11. Data on the non-coincidental exceptional groups

For the *non-coincidental* exceptional irreducible reflection groups W, we tabulate here the polynomials

$$\nu_r(W, q, t) := \frac{\mathrm{Hilb}\left((S(V^*) \otimes \wedge V^* \otimes \wedge^r V)^W, q, t\right)}{\mathrm{Hilb}(S(V^*)^W, q)}$$

for $r = 0, 1, 2, \ldots, n$. The results of Section 3, including Corollary 3.4, 3.16, 3.18, give the same data for the Weyl groups of type A and the infinite family $G(de, e, n)$ of complex reflection groups. Thus, together with Theorem 1.1, this completes those calculations for *all* irreducible complex reflection groups. Also, together with Remark 3.19, it allows one to check that the answers in the non-coincidental cases always differ from what Theorem 1.1 would have predicted.

In this tabulation, we may assume without loss of generality that $1 \le r \le n - 1$, since

$$\nu_0(W, q, t) = \prod_{i=1}^{n}(1 + tq^{e_i}) \quad \text{and} \tag{11.1}$$

$$\nu_n(W, q, t) = \prod_{i=1}^{n}(q^{e_i^*} + t) \tag{11.2}$$

for all reflection groups by Solomon's Theorem [35] and Theorem 2.10, respectively. Additionally, for duality groups W, we may assume $2 \le r \le n - 1$, since equation (2.9) (see [28, eqn. (2.1)]) implies that

$$\nu_1(W, q, t) = \left(\sum_{i=1}^{n} q^{e_i}\right)(1 + tq^{-1}) \prod_{i=1}^{n-1}(1 + tq^{e_i}). \tag{11.3}$$

In general, we will use the notation $[m]_q := 1 + q + q^2 + \cdots + q^{m-1}$.

Rank 2 groups. For rank 2 complex reflection groups, one has a formula for $\nu_1(W,q,t)$ from [28, Cor. 10.2]

$$\nu_1(W,q,t) = (1+tq^{-1})\big((q^{e_1^*}+q^{e_2^*}) + t(q^{e_1+1}+q^{e_2+1})\big).$$

It was noted there that this agrees with equation (11.3), and hence also with Theorem 1.1, exactly when W is a rank 2 duality group, or equivalently, a rank 2 coincidental group.

Real but non-coincidental reflection groups of rank at least 3. For real reflection groups, we may assume that $2 \leq r \leq \lfloor \frac{n}{2} \rfloor$, as they are all duality groups and additionally satisfy (see [28, Prop. 13.1])

$$\nu_{n-r}(W,q,t) = t^n \; \nu_r(W,q,t^{-1})\,.$$

F_4	exponents (1,5,7,11)
ν_2	$(q+t)[2]_{q^4}(1+tq)\cdot$ $\big((q^5+q^7-q^9+q^{11}+q^{13})(1+t^2)+(1+q^6+q^8+q^{10}+q^{12}+q^{18})t\big)$

H_4	exponents (1,11,19,29)
ν_2	$(q+t)(1+tq)\cdot$ $\big((q^{11}+q^{19}+2q^{29}+q^{39}+q^{47})(1+t^2)$ $+(1+q^{10}+q^{18}+q^{20}+q^{22}+q^{28}+q^{30}+q^{36}+q^{38}+q^{40}+q^{48}+q^{58})t\big)$

E_6	exponents (1,4,5,7,8,11)
ν_2	$(q+t)[3]_{q^3}(1+q+q^4+q^7+q^8)\prod_{i=1}^{3}(1+tq^{e_i})\cdot$ $\big(q^4+t\frac{[2]_{q^5}[2]_{q^7}}{[2]_q}+t^2q^7\big)$
ν_3	$[2]_{q^4}[5]_q\prod_{i=1}^{2}(1+tq^{e_i})\prod_{i=1}^{2}(q^{e_i}+t)\cdot$ $\big(q^5(1+t^2)(1-q+q^2+q^3-2q^4+q^5+q^6-q^7+q^8)$ $+t[2]_{q^2}(1-q-q^2+2q^3-q^5+q^6+q^{10}-q^{11}+2q^{13}-q^{14}-q^{15}+q^{16})\big)$

E_7	exponents (1,5,7,9,11,13,17)
ν_2	$(q+t)[3]_{q^6}[7]_{q^2}\prod_{i=1}^{4}(1+tq^{e_i})\cdot$ $\big(q^5+t\frac{[2]_{q^8}[10]_{q^2}}{[2]_{q^2}}+t^2q^{11}\big)$
ν_3	$[5]_{q^2}[7]_{q^2}\prod_{i=1}^{3}(1+tq^{e_i})\prod_{i=1}^{2}(q^{e_i}+t)\cdot$ $\big((q^7+q^9t^2)\frac{[10]_{q^2}}{[2]_{q^2}}+t[2]_{q^4}(1-2q^2+q^4+q^6-q^{10}+q^{14}+q^{16}-2q^{18}+q^{20})\big)$

E_8	exponents (1,7,11,13,17,19,23,29)
ν_2	$(q+t)(1+q^{12})\big(-q^{14}+\sum_{i=1}^{8}q^{e_i-1}\big)\prod_{i=1}^{5}(1+tq^{e_i})\cdot$ $\big((1+q^{12}t^2)((q^7+q^{11})+t(1+q^{14})(1+q^{16})\big)$
ν_3	$[7]_{q^2}\big(\sum_{i=1}^{8}q^{e_i-1}\big)\prod_{i=1}^{4}(1+tq^{e_i})\prod_{i=1}^{2}(q^{e_i}+t)\cdot$ $\big((1+q^6t^2)(q^{11}-q^{15}+q^{17}-q^{19}+q^{23})+t(1-q^2+q^{12}+q^{28}-q^{38}+q^{40})\big)$
ν_4	$[7]_{q^2}\prod_{i=1}^{2}(1+tq^{e_i})\prod_{i=1}^{2}(q^{e_i}+t)\cdot$ $\Big((1+t^4)\cdot q^{24}(1-q^2+q^4+q^6-q^8+q^{10}+q^{12}-2q^{14}+4q^{16}-2q^{20}$ $+4q^{22}-2q^{24}+4q^{28}-2q^{30}+q^{32}+q^{34}-q^{36}+q^{38}+q^{40}-q^{42}+q^{44})$ $+(t+t^3)\cdot q^{11}[2]_{q^6}\big(1-q^4+q^6+3q^{12}-2q^{14}+q^{16}+q^{18}+q^{22}+3q^{24}-q^{26}+q^{28}+2q^{30}$ $+2q^{34}+q^{36}-q^{38}+3q^{40}+q^{42}+q^{46}+q^{48}-2q^{50}+3q^{52}+q^{58}-q^{60}+q^{64}\big)$ $+t^2\cdot(1-q^2+q^6-q^8+q^{10}+q^{12}-q^{14}+q^{16}+q^{18}-q^{20}+3q^{22}+2q^{24}$ $-2q^{26}+5q^{28}+q^{30}+5q^{34}+2q^{36}-q^{38}+9q^{40}-q^{44}+10q^{46}-q^{48}$ $+9q^{52}-q^{54}+2q^{56}+5q^{58}+q^{62}+5q^{64}-2q^{66}+2q^{68}+3q^{70}-q^{72}$ $+q^{74}+q^{76}-q^{78}+q^{80}+q^{82}-q^{84}+q^{86}-q^{90}+q^{92})\Big)$

Duality, but non-real and non-coincidental reflection groups of rank at least 3.

G_{24}	exponents (3,5,13), coexponents (1,9,11)
ν_2	$(q+t)[3]_{q^2}\big((q^6+t^2)(q^3-q^7+q^9)+t(1-q^2+q^6+q^{12})\big)$

G_{27}	exponents (5,11,29), coexponents (1,19,25)
ν_2	$(q+t)\big((q^{19}+q^5t^2)(1+q^6+q^{24})+t(1+q^{18}+2q^{24}+q^{30}+q^{36})\big)$

G_{29}	exponents (3, 7, 11, 19), coexponents (1, 9, 13, 17)
ν_2	$(q+t)(1+q^3t)[3]_{q^4}\big(q^7(1+q^{12})(q^2+t^2)+t(1-q^4+q^8+q^{12}+q^{16}+q^{24})\big)$
ν_3	$\prod_{i=1}^2(q^{e_i^*}+t)\cdot\big((q^{10}+t^2)(\sum_{i=1}^4 q^{e_i^*})+t(1+q^8)(1+q^{12}+2q^{16})\big)$

G_{33}	exponents (3, 5, 9, 11, 17), coexponents (1, 7, 9, 13, 15)
ν_2	$[5]_{q^2}(q+t)\prod_{i=1}^2(1+q^{e_i}t)\cdot$ $\big((q^7+q^9t^2)(1-q^4+q^6+q^8-q^{10}+q^{12})+t(1-q^2+q^6+q^{12}+q^{16}+q^{24})\big)$
ν_3	$[5]_{q^2}(1+q^3t)\prod_{i=1}^2(q^{e_i^*}+t)\cdot$ $\big((q^9+q^5t^2)(1-q^2+q^4+q^6-q^8+q^{12})+t[2]_{q^2}[2]_{q^8}(1-2q^2+2q^4-q^6+q^{10})\big)$
ν_4	$[5]_{q^2}\prod_{i=1}^3(q^{e_i^*}+t)\cdot$ $\big((q^{13}+q^3t^2)(1-q^4+q^6)+t(1-q^2+q^6+q^{16})\big)\big)$

G_{34}	exponents (5, 11, 17, 23, 29, 41), coexponents (1, 13, 19, 25, 31, 37)
ν_2	$[5]_{q^6}(q+t)\prod_{i=1}^3(1+q^{e_i}t)\cdot$ $\big((q^{13}+q^{31}+q^{43})(1+q^{10}t)+t[2]_{q^{12}}(1-q^6+q^{18}+q^{24}+q^{48})\big)$
ν_3	$\begin{bmatrix}5\\2\end{bmatrix}_{q^6}\prod_{i=1}^2(q^{e_i^*}+t)\prod_{i=1}^2(1+q^{e_i}t)\cdot$ $\big(q^{17}+q^{41}(q^2+t^2)+t(1-q^6+q^{18}+q^{24}+q^{36}+q^{48})\big)$
ν_4	$[5]_{q^6}(1+q^5t)\ \prod_{i=1}^3(q^{e_i^*}+t)\cdot$ $\big((q^{11}+q^{23}+q^{41})(q^{14}+t^2)+t(1-q^6+q^{12}+q^{24}+q^{30}+q^{36}+2q^{48})\big)$
ν_5	$[2]_{q^{12}}\prod_{i=1}^4(q^{e_i^*}+t)\cdot$ $\big((q^5+q^{11}+q^{29})(q^{26}+t^2)+t(1+q^{18}+q^{24}+q^{36}+q^{42}+q^{48})\big)$

The unique non-duality exceptional reflection group of rank at least 3.

G_{31}	exponents (7, 11, 19, 23), coexponents (1, 13, 17, 29)
ν_1	$[2]_{q^{12}}(q+t)\prod_{i=1}^2(1+q^{e_i}t)\big(1+q^{16}+t(q^{19}+q^{23})\big)$
ν_2	$(q+t)(1+q^7t)\Big(q^{13}+q^{17}+2q^{29}+q^{41}+q^{45}$ $+t(1+q^{12}+q^{16}+2q^{24}+2q^{28}+2q^{32}+q^{36}+2q^{40})$ $+t^2(q^{11}+q^{19}+2q^{23}+q^{27}+q^{35})\Big)$
ν_3	$[2]_{q^{12}}\prod_{i=1}^3(q^{e_i^*}+t)\big(1+q^{16}+t(q^7+q^{11})\big)$

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