GERSTENHABER BRACKETS FOR SKEW GROUP ALGEBRAS IN POSITIVE CHARACTERISTIC

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Abstract. The deformation theory of an algebra is controlled by the Gerstenhaber bracket, a Lie bracket on Hochschild cohomology. We develop techniques for evaluating Gerstenhaber brackets of semidirect product algebras recording actions of finite groups over fields of positive characteristic. The Hochschild cohomology and Gerstenhaber bracket of these skew group algebras can be complicated when the characteristic of the underlying field divides the group order. We show how to investigate Gerstenhaber brackets using twisted product resolutions, which are often smaller and more convenient than the cumbersome bar resolution typically used. These resolutions provide a concrete description of the Gerstenhaber bracket suitable for exploring questions in deformation theory. We demonstrate with the prototypical example of a graded Hecke algebra (rational Cherednik algebra) in positive characteristic.

1. Introduction

The Hochschild cohomology space of an associative algebra is a Gerstenhaber algebra under two binary operations, the cup product and the Gerstenhaber bracket. The Gerstenhaber bracket is a Lie bracket controlling the deformation theory of the algebra. Historically, it has been more difficult to compute than the cup product: The bracket is defined in terms of the cumbersome bar resolution and notoriously resists transfer to more convenient resolutions. In general, we lack user-friendly formulas giving the Gerstenhaber bracket explicitly.

We consider the Hochschild cohomology of a skew group algebra (semidirect product algebra) arising from the action of a finite group G on an algebra S. We work in the modular setting, i.e., over a field k of positive characteristic that may divide the group order $|G|$. In this setting, the Hochschild cohomology of $S \rtimes G$ is complicated by the potentially onerous cohomology of kG, in contrast to the characteristic zero case where it is always trivial.

Computations of the Gerstenhaber bracket on $S \rtimes G$ directly using the bar resolution often yield little useful information—the bar resolution itself is too large and unwieldy. It can be a struggle even to describe adequately the Hochschild cohomology using the bar resolution. Thus one seeks a description of the Gerstenhaber bracket in terms of smaller

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resolutions used to compute Hochschild cohomology, a description that is concrete and straightforward to apply in specific examples.

In this note, we consider the flexible *twisted product resolution* of a skew group algebra: one chooses a convenient resolution for S and another for G and then combines them to create a resolution of $S \rtimes G$. We show how to apply new techniques from [4] on Gerstenhaber brackets to twisted product resolutions for skew group algebras from [8, 9]. This approach provides advantages over employing the often unmanageable but traditional bar resolution. We produce an explicit description of the Gerstenhaber bracket that should prove user-friendly and we illustrate with an example from deformation theory. This quintessential example using a small transvection group captures the difference between the modular and nonmodular settings, both in the theory of reflection groups and in the theory of graded Hecke algebras (and rational Cherednik algebras, see [3]).

In Section 2, we recall the twisted product resolution from [8, 9] obtained by twisting a resolution of S with one for G. We recall methods of [4] analyzing Gerstenhaber brackets in Section 3 and show how they apply to twisted product resolutions for skew group algebras. We illustrate these techniques by showing how to compute some Gerstenhaber brackets concretely for a small transvection group example from [8] in Section 5. Throughout, k is a field of arbitrary characteristic and $\otimes = \otimes_k$.

2. Twisted product resolutions

We recall the twisted product resolution from [8, 9]. Consider a finite group G acting on a k-algebra S by automorphisms. Let $A = S \rtimes G$ be the corresponding skew group algebra: As a vector space, $S \rtimes G = S \otimes kG$, and we abbreviate the element $s \otimes g$ by sg ($s \in S$, $g \in G$) when no confusion can arise. Multiplication is defined by

$$(sg) \cdot (s'g') = s({}^g s')\, gg' \quad \text{for all } s, s' \in S \text{ and } g, g' \in G\,.$$

The action of g on s' here is denoted by ${}^g s'$. We use the enveloping algebra $S^e = S \otimes S^{op}$ of any algebra S to express bimodule actions as left actions.

The twisted product resolution. We consider projective resolutions

(i) $C: \ldots \to C_2 \to C_1 \to C_0 \to 0$ of kG as a kG-bimodule, and

(ii) $D: \ldots \to D_2 \to D_1 \to D_0 \to 0$ of S as an S-bimodule.

We assume the resolution C is G-graded, with compatible group action:

$$g_1\big((C_i)_{g_2}\big)g_3 = (C_i)_{g_1 g_2 g_3} \quad \text{for all } g_1, g_2, g_3 \in G \text{ and all degrees } i. \tag{2.1}$$

We also assume $D_\bullet$ carries a compatible action of G: Each D_i is left kG-module with

$$g \cdot (s \cdot d) = {}^g s \cdot (g \cdot d) \quad \text{for all } g \in G,\ s \in S,\ d \in D \tag{2.2}$$

and the differentials are kG-module homomorphisms. This ensures $D_\bullet$ is *compatible* with the twisting map $g \otimes s \mapsto {}^g s \otimes g$ given by the group action (see [9, Definition 2.17]). This is the setting, for example, when $C_\bullet$ is the bar or reduced bar resolution of kG and when $D_\bullet$ is the Koszul resolution of a Koszul algebra S (see [9, Prop 2.20(ii)]).

The *twisted product resolution* $X = C \otimes^G D$ of the algebra $S \rtimes G$ is the total complex of the double complex $C_\bullet \otimes D_\bullet$,

$$X = C \otimes^G D \quad \text{where} \quad X_n = \bigoplus_{i+j=n} C_i \otimes D_j$$

with each X_n suffused with the additional structure of a $(S \rtimes G)$-bimodule defined by

$$s'g' \cdot (c \otimes d) \cdot sg = g'cg \ \otimes \ ({}^{(g'hg)^{-1}}s')({}^{g^{-1}}(ds)) \ \text{ for } \ c \in C_h, d \in D, g, g', h \in G, s, s' \in S\,.$$

The differential on X is $\partial_n = \sum_{i+j=n}(\partial_i \otimes 1) + (-1)^i(1 \otimes \partial_j)$ as usual.

With this action, X is a resolution of $A = S \rtimes G$, i.e., X provides an exact sequence of A-bimodules (see [9] or [7, §4]):

$$\ldots \to X_2 \to X_1 \to X_0 \to A \to 0\,.$$

When the A-bimodules X_n are all projective as A^e-modules, X is also a projective resolution of A. This occurs, for example, when D is a Koszul resolution of a Koszul algebra and C is the bar resolution of kG. (See [9, Proposition 2.20(ii)].)

3. Gerstenhaber brackets on differential graded coalgebras

In this section, we summarize some results of [4] and develop additional techniques for computing Gerstenhaber brackets in the modular setting. Contrast with [5, 6], where the characteristic of the underlying field was 0.

Resolutions as differential graded coalgebras. Consider a k-algebra A and a projective resolution P of A as an A-bimodule:

$$\ldots \to P_2 \to P_1 \to P_0 \to 0\,.$$

The resolution P is a *differential graded coalgebra* when $P = \oplus_i P_i$ has a coalgebra structure compatible with its differential ∂_P. This means there is a (degree 0) chain map $\Delta_P : P \to P \otimes_A P$ lifting the canonical isomorphism $A \xrightarrow{\sim} A \otimes_A A$, called a *diagonal map*, that is required to be

coassociative, i.e., $(\Delta_P \otimes 1)\Delta_P = (1 \otimes \Delta_P)\Delta_P$ as maps $P \to P \otimes_A P \otimes_A P$, and
counital, i.e., $(\mu_P \otimes 1_P)\Delta_P = 1_P = (1_P \otimes \mu_P)\Delta_P$ as maps $P \to P$,

where $\mu_P : P_0 \to A$ is augmentation of the complex (with μ_P zero on P_i for $i \geq 1$). Throughout, we define $\mu_P \otimes 1_P : P \otimes P \to P$ as the map $p \otimes p' \mapsto \mu_P(p) \cdot p'$ (and similarly for $1_P \otimes \mu_P$). Recall that the differential on $P_n \otimes_A P_m$ is just $\partial_P \otimes 1_P + (-1)^n 1_P \otimes \partial_P$.

Homotopy from right to left. We may map the complex $P \otimes_A P$ to the complex P using either $\mu_P \otimes 1_P$ or $1_P \otimes \mu_P$. When P is a differential graded coalgebra, these mappings are chain homotopic by [4, Lemma 3.2.1]. (The hypotheses there are slightly stronger, but the same proof works under our hypotheses here.) Thus there exists a chain homotopy from $\mu_P \otimes 1_P$ to $1_P \otimes \mu_P$, i.e., a map $\phi_P : P \otimes_A P \to P$ with $P_m \otimes_A P_n \to P_{m+n+1}$ satisfying

$$\partial_P\, \phi_P + \phi_P\, \partial_{P \otimes_A P} = \mu_P \otimes 1_P - 1_P \otimes \mu_P\,. \tag{3.1}$$

Example 3.2. The bar resolution B of the algebra A is a differential graded coalgebra. Indeed, for $B_n = A \otimes A^{\otimes n} \otimes A$, a diagonal map $\Delta_B : B \to B \otimes_A B$ is defined by

$$\Delta_B(a_0 \otimes \cdots \otimes a_{n+1}) = \sum_{j=0}^{n} (a_0 \otimes \cdots \otimes a_j \otimes 1) \otimes_A (1 \otimes a_{j+1} \otimes \cdots \otimes a_{n+1}) \tag{3.3}$$

for $a_0, \dots, a_{n+1}$ in A. This map is coassociative and counital. One choice of homotopy $\phi_B : B \otimes_A B \to B$ from $\mu_B \otimes 1_B$ to $1_B \otimes \mu_B$ is defined by

$$\begin{aligned} &\phi_B\big((a_0 \otimes \cdots \otimes a_{p-1} \otimes a_p) \otimes_A (a'_p \otimes a_{p+1} \otimes \cdots \otimes a_{n+1})\big) \\ &\quad = (-1)^{p-1}\, a_0 \otimes \cdots \otimes a_{p-1} \otimes a_p a'_p \otimes a_{p+1} \otimes \cdots \otimes a_{n+1} \qquad \text{for all } a_i, a'_p \in A\,. \end{aligned} \tag{3.4}$$

Koszul resolutions of Koszul algebras are also differential graded coalgebras [1]. The Koszul resolution P of a Koszul algebra embeds into the bar resolution, however the above map ϕ_B does not preserve the image. Instead, a homotopy ϕ_P may be found directly in this case; see [4, §4], [5, §3.2], or [2, §4] for some examples.

Definition of the Gerstenhaber bracket. The Gerstenhaber bracket for A is defined on cochains on the bar resolution B of A. Identify each space of cochains $\mathrm{Hom}_{A^e}(B_n, A)$ with $\mathrm{Hom}_k(A^{\otimes n}, A)$ via the canonical isomorphism. Then the Gerstenhaber bracket

$$[\ ,\ ] : \mathrm{Hom}_k(A^{\otimes n}, A) \times \mathrm{Hom}_k(A^{\otimes m}, A) \to \mathrm{Hom}_k(A^{\otimes (n+m-1)}, A)$$

on cochains is defined by

$$[f, f'] \ = \ f \circ f' - (-1)^{(n-1)(m-1)} f' \circ f$$

where, for a_i in A, the circle product $(f \circ f')(a_1 \otimes \cdots \otimes a_{n+m-1})$ is

$$\sum_{i=1}^{n} (-1)^{(m-1)(i-1)}\, f\Big(a_1 \otimes \cdots \otimes a_{i-1} \otimes f'(a_i \otimes \cdots \otimes a_{i+m-1}) \otimes a_{i+m} \otimes \cdots \otimes a_{n+m-1}\Big)\,.$$

Gerstenhaber brackets on differential graded coalgebras. Although the Gerstenhaber bracket is defined using the bar resolution, we seek descriptions in terms of more convenient resolutions used to compute Hochschild cohomology. Suppose P is a projective resolution of A with a differential graded coalgebra structure. The Gerstenhaber bracket can be defined directly at the chain level on P using [4, Theorem 3.2.5]; we recall how a homotopy ϕ_P (see (3.1)) gives the bracket explicitly.

Extend any cochain $f \in \mathrm{Hom}_{A^e}(P_n, A)$ to all of P by defining $f \equiv 0$ on P_m with $m \neq n$. For $f \in \mathrm{Hom}_{A^e}(P_n, A)$ and $f' \in \mathrm{Hom}_{A^e}(P_m, A)$, define

$$[f, f']_P = f \circ_P f' - (-1)^{(n-1)(m-1)} f' \circ_P f \tag{3.5}$$

where $f \circ_P f'$ (similarly $f' \circ_P f$) is the composition

$$f \circ_P f' : P \xrightarrow{\Delta_P^{(2)}} P \otimes_A P \otimes_A P \xrightarrow{1_P \otimes f' \otimes 1_P} P \otimes_A P \xrightarrow{\phi_P} P \xrightarrow{f} A. \tag{3.6}$$

Here, $\Delta_P^{(2)} = (1_P \otimes \Delta_P)\Delta_P = (\Delta_P \otimes 1_P)\Delta_P$ and $1_P \otimes f' \otimes 1_P$ has signs attached so that

$$(1_P \otimes f' \otimes 1_P)(x \otimes y \otimes z) = (-1)^{im} x \otimes f'(y) \otimes z \tag{3.7}$$

for $x \in P_i$, $y, z \in P$. Then [4, Theorem 3.2.5] implies that the Gerstenhaber bracket $[\ ,\ ]$ of any elements in cohomology is given at the cochain level on P by the map $[\ ,\ ]_P$ on cocycles. (Note that [4, Theorem 3.2.5] has slightly stronger hypotheses, but the proof indeed holds for any resolution P with the structure of a differential graded coalgebra.)

4. Twisted product resolution as a differential graded coalgebra

We show in this section that a twisted product resolution X of $S \rtimes G$ constructed from two differential graded coalgebras C and D is again a differential graded coalgebra. We then give the Gerstenhaber bracket for X in terms of the maps describing the Gerstenhaber brackets of C and D individually.

Throughout this section, we fix

- a differential graded coalgebra bimodule resolution (C, Δ_C, μ_C) of G and
- a differential graded coalgebra bimodule resolution (D, Δ_D, μ_D) of S, producing
- a twisted product resolution $X = C \otimes^G D$ of $A = S \rtimes G$.

We assume that C is G-graded (as in (2.1)) with Δ_C, μ_C preserving the grading and also that D carries a G-action (as in (2.2)) with Δ_D, μ_D both kG-module homomorphisms. This is the case, for example, if C is the bar (or reduced bar) resolution of kG and D is the Koszul resolution of a Koszul algebra (see [9, Proposition 2.20(ii)]).

Twisted comultiplication. In the next lemmas, we use diagonal maps for C and D to produce a diagonal map $\Delta_X : X \to X \otimes_A X$.

Lemma 4.1. *Define a twisting map $\tau : C \otimes D \to D \otimes C$ by*

$$\tau_{i,j}(c \otimes d) = (-1)^{ij}({}^g d) \otimes c \quad \text{for all } c \in (C_i)_g \text{ and } d \in D_j \,. \tag{4.2}$$

Then τ extends to a well-defined chain map

$$1 \otimes \tau \otimes 1 : (C \otimes_{kG} C) \otimes (D \otimes_S D) \longrightarrow (C \otimes^G D) \otimes_{S \rtimes G} (C \otimes^G D) \,.$$

Proof. Consider the map

$$C \otimes C \otimes D \otimes D \xrightarrow{1 \otimes \tau \otimes 1} C \otimes D \otimes C \otimes D \longrightarrow (C \otimes^G D) \otimes_{S \rtimes G} (C \otimes^G D),$$

where the latter map is the canonical surjection. Calculations show that the composition of these two maps is kG-middle linear in the first two arguments and S-middle linear in the last two arguments, and so it induces a well-defined map as claimed. A calculation shows that it is a chain map. □

Lemma 4.3. *Let $X = C \otimes^G D$ be a twisted product resolution of $S \rtimes G$ for differential graded coalgebras C and D resolving kG and S, respectively, as above. Then X is a differential graded coalgebra as well with comultiplication $\Delta_X : X \to X \otimes_A X$ given by*

$$\Delta_X = (1 \otimes \tau \otimes 1)(\Delta_C \otimes \Delta_D) \,.$$

Proof. We first check that Δ_X is coassociative using the fact that Δ_C and Δ_D are each coassociative. We use the G-grading on C and the compatible G-action on D:

$$\begin{aligned}
&(\Delta_X \otimes 1_X)\Delta_X \\
&= \big((1\otimes\tau\otimes 1)(\Delta_C\otimes\Delta_D)\otimes 1\otimes 1\big)(1\otimes\tau\otimes 1)(\Delta_C\otimes\Delta_D) \\
&= (1\otimes\tau\otimes 1\otimes 1\otimes 1)(\Delta_C\otimes\Delta_D\otimes 1\otimes 1)(1\otimes\tau\otimes 1)(\Delta_C\otimes\Delta_D) \\
&= (1\otimes\tau\otimes 1\otimes 1\otimes 1)(1\otimes 1\otimes 1\otimes\tau\otimes 1)(1\otimes 1\otimes\tau\otimes 1\otimes 1)(\Delta_C\otimes 1\otimes\Delta_D\otimes 1)(\Delta_C\otimes\Delta_D) \\
&= (1\otimes\tau\otimes 1\otimes 1\otimes 1)(1\otimes 1\otimes 1\otimes\tau\otimes 1)(1\otimes 1\otimes\tau\otimes 1\otimes 1)(1\otimes\Delta_C\otimes 1\otimes\Delta_D)(\Delta_C\otimes\Delta_D) \\
&= (1\otimes 1\otimes 1\otimes\tau\otimes 1)(1\otimes\tau\otimes 1\otimes 1\otimes 1)(1\otimes 1\otimes\tau\otimes 1\otimes 1)(1\otimes\Delta_C\otimes 1\otimes\Delta_D)(\Delta_C\otimes\Delta_D) \\
&= (1\otimes 1\otimes 1\otimes\tau\otimes 1)(1\otimes 1\otimes\Delta_C\otimes\Delta_D)(1\otimes\tau\otimes 1)(\Delta_C\otimes\Delta_D) \\
&= (1_X\otimes\Delta_X)\Delta_X .
\end{aligned}$$

We next verify that Δ_X is counital using the fact that Δ_C and Δ_D are each counital. We use the extra assumption that μ_C preserves the G-grading and μ_D is a kG-module homomorphism as well as the definition of the $S \rtimes G$-bimodule structure on $C \otimes^G D$:

$$\begin{aligned}
(\mu_X\otimes 1_X)\Delta_X &= (\mu_C\otimes\mu_D\otimes 1\otimes 1)(1\otimes\tau\otimes 1)(\Delta_C\otimes\Delta_D) \\
&= (\mu_C\otimes 1\otimes\mu_D\otimes 1)(\Delta_C\otimes\Delta_D) = ((\mu_C\otimes 1)\Delta_C)\otimes((\mu_D\otimes 1)\Delta_D) \\
&= 1\otimes 1 \quad = \quad 1_X,
\end{aligned}$$

and, similarly, $(1_X\otimes\mu_X)\Delta_X = 1_X$.

We now need only check that Δ_X is a chain map, i.e., $\Delta_X\,\partial = (\partial\otimes 1 + 1\otimes\partial)\Delta_X$, for ∂ the differential on X. This follows from the fact that τ, Δ_C, Δ_D are all chain maps. □

Remark 4.4. One may check that the map $1\otimes\tau\otimes 1$ of (4.2) interpolates between the maps of the form $\mu\otimes 1 - 1\otimes\mu$ for the various complexes, that is,

$$\mu_X\otimes 1_X - 1_X\otimes\mu_X = (\mu_C\otimes 1_C\otimes\mu_D\otimes 1_D - 1_C\otimes\mu_C\otimes 1_D\otimes\mu_D)(1_C\otimes\tau^{-1}\otimes 1_D). \tag{4.5}$$

We now give a theorem describing a homotopy from $\mu_X\otimes 1_X$ to $1_X\otimes\mu_X$ concretely in terms of homotopies from $\mu_C\otimes 1_C$ to $1_C\otimes\mu_C$ and from $\mu_D\otimes 1_D$ to $1_D\otimes\mu_D$ by adapting [2, Lemmas 3.3, 3.4, and 3.5] to our setting.

Theorem 4.6. *Let $X = C\otimes^G D$ as above with homotopies ϕ_C from $\mu_C\otimes 1_C$ to $1_C\otimes\mu_C$ and ϕ_D from $\mu_D\otimes 1_D$ to $1_D\otimes\mu_D$. Define $\phi_X : X\otimes_A X\to X$ by*

$$\phi_X = \big(\phi_C\otimes\mu_D\otimes 1_D + \epsilon_C(1_C\otimes\mu_C)\otimes\phi_D\big)(1_C\otimes\tau^{-1}\otimes 1_D)$$

for $\epsilon_C : C\to C$ defined by $c\mapsto(-1)^{|c|}$ for homogeneous c. Then ϕ_X is a homotopy from $\mu_X\otimes 1_X$ to $1_X\otimes\mu_X$.

Proof. Let $\phi'_X : C\otimes C\otimes D\otimes D\to C\otimes D$ be the map $\phi_C\otimes\mu_D\otimes 1 + \epsilon_C(1\otimes\mu_C)\otimes\phi_D$ so that $\phi_X = \phi'_X(1\otimes\tau^{-1}\otimes 1)$. Then on $(C\otimes C)\otimes(D\otimes D)$,

$$\begin{aligned}
\partial_X\,\phi_X\,(1\otimes\tau\otimes 1) &= \partial_X\,\phi'_X \\
&= (\partial_C\otimes 1 + \epsilon_C\otimes\partial_D)\big(\phi_C\otimes\mu_D\otimes 1 + \epsilon_C(1\otimes\mu_C)\otimes\phi_D\big) \\
&= \partial_C\phi_C\otimes\mu_D\otimes 1 - \phi_C(\epsilon_C\otimes\epsilon_C)\otimes\partial_D(\mu_D\otimes 1) \\
&\qquad - \epsilon_C\partial_C(1\otimes\mu_C)\otimes\phi_D + 1\otimes\mu_C\otimes\partial_D\phi_D\,,
\end{aligned} \tag{4.7}$$

and, since $1 \otimes \tau \otimes 1$ is a chain map from $(C \otimes C) \otimes (D \otimes D)$ to $X \otimes_A X$,

$$
\begin{aligned}
\phi_X\, \partial_{X\otimes X}\, (1 \otimes \tau \otimes 1) &= \phi_X\, (1 \otimes \tau \otimes 1)\, \partial_{(C\otimes C)\otimes(D\otimes D)} = \phi'_X\ \partial_{(C\otimes C)\otimes(D\otimes D)} \\
&= \phi'_X \left(\partial_{C\otimes C} \otimes 1_{D\otimes D} + (\epsilon_C \otimes \epsilon_C) \otimes \partial_{D\otimes D}\right) \\
&= \phi_C \partial_{C\otimes C} \otimes \mu_D \otimes 1 + \phi_C(\epsilon_C \otimes \epsilon_C) \otimes (\mu_D \otimes 1)\partial_{D\otimes D} \\
&\quad + \epsilon_C(1 \otimes \mu_C)\partial_{C\otimes C} \otimes \phi_D + (1 \otimes \mu_C) \otimes \phi_D \partial_{D\otimes D}.
\end{aligned} \tag{4.8}
$$

Here we used the fact that $\epsilon_C\phi_C = -\phi_C(\epsilon_C \otimes \epsilon_C)$, $\epsilon_C(1 \otimes \mu_C)(\epsilon_C \otimes \epsilon_C) = 1 \otimes \mu_C$, and $\partial_C\epsilon_C = -\epsilon_C\partial_C$. The second term of (4.7) cancels with the second term of (4.8) as $\mu_D \otimes 1$ is a chain map; likewise, the third terms cancel as $\mu_C \otimes 1$ is a chain map. Hence

$$
\begin{aligned}
(\partial_X\, \phi_X\ &+ \phi_X\, \partial_{X\otimes X})(1 \otimes \tau \otimes 1) \\
&= (\partial\phi_C + \phi_C\partial) \otimes \mu_D \otimes 1 + 1 \otimes \mu_C \otimes (\partial\phi_D + \phi_D\partial) \\
&= (\mu_C \otimes 1 - 1 \otimes \mu_C) \otimes \mu_D \otimes 1 + 1 \otimes \mu_C \otimes (\mu_D \otimes 1 - 1 \otimes \mu_D) \\
&= \mu_C \otimes 1 \otimes \mu_D \otimes 1 - 1 \otimes \mu_C \otimes 1 \otimes \mu_D,
\end{aligned}
$$

and, by equation (4.5),

$$
\partial\phi_X + \phi_X\partial = (\mu_C \otimes 1 \otimes \mu_D \otimes 1 - 1 \otimes \mu_C \otimes 1 \otimes \mu_D)(1 \otimes \tau^{-1} \otimes 1) = \mu_X \otimes 1 - 1 \otimes \mu_X.
$$

□

Gerstenhaber bracket for skew group algebras. The next theorem gives the Gerstenhaber bracket on a twisted product resolution X. Note that the twisting map τ in the theorem is from Lemma 4.1, the map $1_X \otimes f' \otimes 1_X$ has signs attached as in (3.7), and ϵ_C merely adjusts signs, $c \mapsto (-1)^{|c|}$ for homogeneous c in C.

Theorem 4.9. *Let $X = C \otimes^G D$ be a twisted product resolution of $S \rtimes G$ for differential graded coalgebras (C, Δ_C, μ_C) and (D, Δ_D, μ_D) resolving kG and S, respectively, as above. The Gerstenhaber bracket of elements of Hochschild cohomology represented by cocycles $f \in \operatorname{Hom}_{A^e}(X_n, A)$ and $f' \in \operatorname{Hom}_{A^e}(X_m, A)$ is represented by the cocycle*

$$
[f, f'] = f \circ_X f' - (-1)^{(n-1)(m-1)} f' \circ_X f\,, \tag{4.10}
$$

where $f \circ_X f'$ (similarly $f' \circ_X f$) is the composition

$$
X \xrightarrow{(1_X\otimes\Delta_X)(\Delta_X)} X \otimes_A X \otimes_A X \xrightarrow{1_X\otimes f'\otimes 1_X} X \otimes_A X \xrightarrow{\phi_X} X \xrightarrow{f} A \tag{4.11}
$$

with

$$
\begin{aligned}
\Delta_X &= (1_C \otimes \tau \otimes 1_D)(\Delta_C \otimes \Delta_D)\,, \ \textit{and} \\
\phi_X &= \left(\phi_C \otimes \mu_D \otimes 1_D + (1 \otimes \mu_C)(\epsilon_C \otimes 1) \otimes \phi_D\right)(1 \otimes \tau^{-1} \otimes 1)\,.
\end{aligned}
$$

Proof. We combine Lemmas 4.1, Lemma 4.3, and Theorem 4.6 with (3.6) and (3.5). □

Example 4.12. In case $S = S(V) \cong k[x_1, \ldots, x_n]$, the symmetric algebra on a finite dimensional vector space V, we take D to be the Koszul resolution for which a choice of ϕ_D has been made in [4, §4] (see also [5, §3.2]). We may take C to be the bar or reduced bar resolution of kG for some applications, with homotopy ϕ_C as defined by equation (3.4).

5. A small transvection group example

We end by demonstrating how to use a twisted product resolution to compute Gerstenhaber brackets explicitly via Theorem 4.9. We also see how computation of explicit brackets can shed light on questions in deformation theory (see [8]). We illustrate with the prototype example of a graded Hecke algebra (or rational Cherednik algebra) in positive characteristic (see [3] and [8]). In the nonmodular setting, these algebras have parameters supported only on the identity group element and on bireflections; in the modular setting, parameters can also be supported on reflections. All reflections in a finite linear group G acting in the modular setting are either diagonalizable or act as in this example. We include some explicit details to illustrate how to evaluate the maps in Theorem 4.9 concretely. We find both a nonzero and a zero Gerstenhaber bracket.

5.1. **Group action and twisted product resolution.** Say $\mathrm{char}(k) = p > 0$ and consider the cyclic group $G \simeq \mathbb{Z}/p\mathbb{Z}$ acting on $V = k^2$ with basis v, w generated by

$$g = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right), \quad \text{so that} \quad {}^g v = v \ \text{ and } \ {}^g w = v + w.$$

We work in the twisted product resolution $X = C \otimes^G D$ of $S(V) \rtimes G$ obtained from twisting the reduced bar resolution C of kG with the Koszul resolution D of $S(V)$:

$$X_n = \bigoplus_{i+j=n} X_{i,j} \quad \text{for} \quad X_{i,j} = kG \otimes (\overline{kG})^{\otimes i} \otimes kG \otimes S(V) \otimes \textstyle\bigwedge^j V \otimes S(V).$$

Here, $C_n = kG \otimes (\overline{kG})^{\otimes n} \otimes kG$ with $\overline{kG} = kG/k1_G$ and $D_n = S(V) \otimes \bigwedge^n V \otimes S(V)$. Then C and D satisfy the conditions specified in Section 3, and Theorem 4.9 applies.

5.2. **Cochains.** Consider cochains on the resolution X:

$$\begin{aligned} &\kappa \in \mathrm{Hom}_{(S(V)\rtimes G)^e}(X_{0,2}, S(V) \rtimes G), \\ &\lambda \in \mathrm{Hom}_{(S(V)\rtimes G)^e}(X_{1,1}, S(V) \rtimes G), \text{ and} \\ &\delta \in \mathrm{Hom}_{(S(V)\rtimes G)^e}(X_{0,1}, S(V) \rtimes G) \end{aligned}$$

defined by (with subscripts on the tensor signs suppressed for brevity)

$$\begin{aligned} \lambda\big((1_G \otimes g^i \otimes 1_G) \otimes (1_S \otimes v \otimes 1_S)\big) &= 0, \\ \lambda\big((1_G \otimes g^i \otimes 1_G) \otimes (1_S \otimes w \otimes 1_S)\big) &= i g^{i-1}, \\ \kappa\big((1_G \otimes 1_G) \otimes (1_S \otimes v \wedge w \otimes 1_S)\big) &= g, \\ \delta\big((1_G \otimes 1_G) \otimes (1_S \otimes v \otimes 1_S)\big) &= v, \\ \delta\big((1_G \otimes 1_G) \otimes (1_S \otimes w \otimes 1_S)\big) &= 0 \end{aligned}$$

for $0 \le i \le p-1$, with all other values determined by these. One can check directly that λ and κ are 2-cocycles and that δ is a 1-cocycle for X. We will show that

$$[\delta, \kappa] \neq 0 \quad \text{and} \quad [\lambda, \lambda] = [\lambda, \kappa] = 0\,.$$

The diagonal maps. We give some values of the diagonal maps at play in finding the Gerstenhaber brackets. The diagonal map Δ_C on the reduced bar resolution of kG is deduced from (3.3). For example, after identifying g^i with its image in $\overline{kG}$,

$$\Delta_C(1_G \otimes g^i \otimes 1_G) = (1_G \otimes 1_G) \otimes_{kG} (1_G \otimes g^i \otimes 1_G) + (1_G \otimes g^i \otimes 1_G) \otimes_{kG} (1_G \otimes 1_G), \quad \text{and}$$
$$\Delta_C(1_G \otimes 1_G) = (1_G \otimes 1_G) \otimes_{kG} (1_G \otimes 1_G)\,.$$

The diagonal map Δ_D is found from embedding the Koszul into the bar resolution and then using (3.3). For example, we identify $v \wedge w$ with $v \otimes w - w \otimes v$ and observe that

$$\begin{aligned}
\Delta_D(1_S \otimes v \wedge w \otimes 1_S) \;&=\; (1_S \otimes 1_S) \otimes_S (1_S \otimes v \wedge w \otimes 1_S) \\
&\quad + (1_S \otimes v \otimes 1_S) \otimes_S (1 \otimes w \otimes 1) - (1_S \otimes w \otimes 1_S) \otimes_S (1_S \otimes v \otimes 1_S) \\
&\quad + (1_S \otimes v \wedge w \otimes 1_S) \otimes_S (1_S \otimes 1_S)\,, \quad \text{and} \\
\Delta_D(1_S \otimes v \otimes 1_S) \;&=\; (1_S \otimes 1_S) \otimes_S (1_S \otimes v \otimes 1_S) + (1_S \otimes v \otimes 1_S) \otimes_S (1_S \otimes 1_S)\,.
\end{aligned}$$

Homotopies. Let $\phi_C : C \otimes_{kG} C \to C$ be the homotopy from $\mu_C \otimes 1$ to $1 \otimes \mu_C$ from (3.4). We choose the homotopy $\phi_D : D \otimes_S D \to D$ from $\mu_D \otimes 1$ to $1 \otimes \mu_D$ given in [4, Definition 4.1.3] and record a few values here for later use:

$$\phi_D((1 \otimes w \otimes 1) \otimes_S (v \otimes 1)) = 1 \otimes v \wedge w \otimes 1, \qquad \phi_D((1 \otimes v) \otimes_S (1 \otimes w \otimes 1)) = 0,$$
$$\phi_D((1 \otimes 1) \otimes_S (1 \otimes v \otimes 1)) = 0, \qquad \phi_D((1 \otimes v \otimes 1) \otimes_S (1 \otimes 1)) = 0\,.$$

Nonzero bracket. We use Theorem 4.9 to show explicitly that $[\delta, \kappa] = \kappa$. First note that $[\delta, \kappa]$ is zero on all components of X except possibly $X_{0,2}$. We consider the composition (4.11) with $f' = \delta$ and $f = \kappa$ to find $\kappa \circ_X \delta$. As a first step, we apply the map $(\Delta_X \otimes 1_X)\Delta_X$ to the element $(1_G \otimes 1_G) \otimes (1_S \otimes v \wedge w \otimes 1_S)$ of $X_{0,2}$, where, recall

$$\Delta_X = (1_C \otimes \tau \otimes 1_D)(\Delta_C \otimes \Delta_D)\,.$$

Direct calculation confirms that

$$\begin{aligned}
&(1 \otimes 1) \otimes (1 \otimes v \wedge w \otimes 1) \xmapsto{(\Delta_X \otimes 1)\Delta_X} \\
&\qquad (1 \otimes 1) \otimes (1 \otimes 1) \otimes (1 \otimes 1) \otimes (1 \otimes 1) \otimes (1 \otimes 1) \otimes (1 \otimes v \wedge w \otimes 1) \\
&\qquad + (1 \otimes 1) \otimes (1 \otimes 1) \otimes (1 \otimes 1) \otimes (1 \otimes v \otimes 1) \otimes (1 \otimes 1) \otimes (1 \otimes w \otimes 1) \\
&\qquad + (1 \otimes 1) \otimes (1 \otimes v \otimes 1) \otimes (1 \otimes 1) \otimes (1 \otimes 1) \otimes (1 \otimes 1) \otimes (1 \otimes w \otimes 1) \\
&\qquad - (1 \otimes 1) \otimes (1 \otimes 1) \otimes (1 \otimes 1) \otimes (1 \otimes w \otimes 1) \otimes (1 \otimes 1) \otimes (1 \otimes v \otimes 1) \\
&\qquad - (1 \otimes 1) \otimes (1 \otimes w \otimes 1) \otimes (1 \otimes 1) \otimes (1 \otimes 1) \otimes (1 \otimes 1) \otimes (1 \otimes v \otimes 1) \\
&\qquad + (1 \otimes 1) \otimes (1 \otimes 1) \otimes (1 \otimes 1) \otimes (1 \otimes v \wedge w \otimes 1) \otimes (1 \otimes 1) \otimes (1 \otimes 1) \\
&\qquad + (1 \otimes 1) \otimes (1 \otimes v \otimes 1) \otimes (1 \otimes 1) \otimes (1 \otimes w \otimes 1) \otimes (1 \otimes 1) \otimes (1 \otimes 1) \\
&\qquad - (1 \otimes 1) \otimes (1 \otimes w \otimes 1) \otimes (1 \otimes 1) \otimes (1 \otimes v \otimes 1) \otimes (1 \otimes 1) \otimes (1 \otimes 1) \\
&\qquad + (1 \otimes 1) \otimes (1 \otimes v \wedge w \otimes 1) \otimes (1 \otimes 1) \otimes (1 \otimes 1) \otimes (1 \otimes 1) \otimes (1 \otimes 1)
\end{aligned}$$

as an element of $X \otimes_A X \otimes_A X$. We have suppressed all subscripts for brevity; for example, the second summand may be written

$$\big((1_G \otimes_{kG} 1_G) \otimes (1_S \otimes_S 1_S)\big) \otimes_A \big((1_G \otimes_{kG} 1_G) \otimes (1_S \otimes_S v \otimes_S 1_S)\big) \otimes_A \big((1_G \otimes_{kG} 1_G) \otimes (1_S \otimes_S w \otimes_S 1_S)\big)\,.$$

We next apply the map $1_X \otimes \delta \otimes 1_X$; it is nonzero on exactly two summands, the second and the penultimate, and we obtain (with the tensor products over A indicated here)

$$\big((1_G \otimes 1_G) \otimes (1_S \otimes v)\big) \otimes_A \big((1_G \otimes 1_G) \otimes (1_S \otimes w \otimes 1_S)\big) - \big((1_G \otimes 1_G) \otimes (1_S \otimes w \otimes v)\big) \otimes_A \big((1_G \otimes 1_G) \otimes (1_S \otimes 1_S)\big).$$

To apply ϕ_X next, we first rearrange terms with $1_G \otimes \tau^{-1} \otimes 1_S$, producing

$$(1_G\otimes 1_G)\otimes(1_G\otimes 1_G)\otimes(1_S\otimes v)\otimes(1_S\otimes w\otimes 1_S)-(1_G\otimes 1_G)\otimes(1_G\otimes 1_G)\otimes(1_S\otimes w\otimes v)\otimes(1_S\otimes 1_S),$$

and then apply the map $\phi_C \otimes \mu_D \otimes 1_D + 1_C \otimes \mu_C \otimes \phi_D$ to obtain

$$(1_G \otimes 1_G \otimes 1_G) \otimes (v \otimes w \otimes 1_S) - (1_G \otimes 1_G) \otimes (1_S \otimes v \wedge w \otimes 1_S).$$

Lastly, we apply κ as the last step of (4.11) and obtain 0 from the first term and $-g$ from the second. Thus

$$(\kappa \circ \delta)\big((1_G \otimes 1_G) \otimes (1_S \otimes v \wedge w \otimes 1_S)\big) = -g = \kappa\big((1_G \otimes 1_G) \otimes (1_S \otimes v \wedge w \otimes 1_S)\big)$$

and $\kappa \circ_X \delta = -\kappa$. We inspect the above calculation with an eye toward switching the order of κ and δ and deduce that $\delta \circ_X \kappa = 0$. We conclude, as claimed,

$$[\delta, \kappa] = \delta \circ_X \kappa - \kappa \circ_X \delta = \kappa\,.$$

Zero brackets. We now use Theorem 4.9 to show that $[\lambda, f] = 0$ when f is λ or κ. We evaluate composition (4.11) on $X_{1,2}$ with $f' = \lambda$. Other calculations are similar. We first apply $\Delta_X = (1_G \otimes \tau \otimes 1_S)(\Delta_C \otimes \Delta_D)$ to sample input in $X_{1,2}$, noting that ${}^{g^i}w = iv + w$ (with subscripts suppressed again):

$$\begin{aligned}
&(1 \otimes g^i \otimes 1) \otimes (1 \otimes v \wedge w \otimes 1)\\
&\qquad\xmapsto{\Delta_C\otimes\Delta_D} (1 \otimes 1) \otimes (1 \otimes g^i \otimes 1) \otimes (1 \otimes 1) \otimes (1 \otimes v \wedge w \otimes 1)\\
&\qquad\qquad +(1 \otimes 1) \otimes (1 \otimes g^i \otimes 1) \otimes (1 \otimes v \otimes 1) \otimes (1 \otimes w \otimes 1)\\
&\qquad\qquad -(1 \otimes 1) \otimes (1 \otimes g^i \otimes 1) \otimes (1 \otimes w \otimes 1) \otimes (1 \otimes v \otimes 1)\\
&\qquad\qquad +(1 \otimes 1) \otimes (1 \otimes g^i \otimes 1) \otimes (1 \otimes v \wedge w \otimes 1) \otimes (1 \otimes 1)\\
&\qquad\qquad +(1 \otimes g^i \otimes 1) \otimes (1 \otimes 1) \otimes (1 \otimes 1) \otimes (1 \otimes v \wedge w \otimes 1)\\
&\qquad\qquad +(1 \otimes g^i \otimes 1) \otimes (1 \otimes 1) \otimes (1 \otimes v \otimes 1) \otimes (1 \otimes w \otimes 1)\\
&\qquad\qquad -(1 \otimes g^i \otimes 1) \otimes (1 \otimes 1) \otimes (1 \otimes w \otimes 1) \otimes (1 \otimes v \otimes 1)\\
&\qquad\qquad +(1 \otimes g^i \otimes 1) \otimes (1 \otimes 1) \otimes (1 \otimes v \wedge w \otimes 1) \otimes (1 \otimes 1)\\
&\qquad\xmapsto{1\otimes\tau\otimes 1} (1 \otimes 1) \otimes (1 \otimes 1) \otimes (1 \otimes g^i \otimes 1) \otimes (1 \otimes v \wedge w \otimes 1)\\
&\qquad\qquad -(1 \otimes 1) \otimes (1 \otimes v \otimes 1) \otimes (1 \otimes g^i \otimes 1) \otimes (1 \otimes w \otimes 1)\\
&\qquad\qquad +(1 \otimes 1) \otimes (1 \otimes (iv + w) \otimes 1) \otimes (1 \otimes g^i \otimes 1) \otimes (1 \otimes v \otimes 1)\\
&\qquad\qquad +(1 \otimes 1) \otimes (1 \otimes v \wedge w \otimes 1) \otimes (1 \otimes g^i \otimes 1) \otimes (1 \otimes 1)\\
&\qquad\qquad +(1 \otimes g^i \otimes 1) \otimes (1 \otimes 1) \otimes (1 \otimes 1) \otimes (1 \otimes v \wedge w \otimes 1)\\
&\qquad\qquad +(1 \otimes g^i \otimes 1) \otimes (1 \otimes v \otimes 1) \otimes (1 \otimes 1) \otimes (1 \otimes w \otimes 1)\\
&\qquad\qquad -(1 \otimes g^i \otimes 1) \otimes (1 \otimes w \otimes 1) \otimes (1 \otimes 1) \otimes (1 \otimes v \otimes 1)\\
&\qquad\qquad +(1 \otimes g^i \otimes 1) \otimes (1 \otimes v \wedge w \otimes 1) \otimes (1 \otimes 1) \otimes (1 \otimes 1)\,,
\end{aligned}$$

an element of $X \otimes_A X$. Next we apply $\Delta_X \otimes 1_X$: Evaluating $\Delta_C \otimes \Delta_D \otimes 1_X$ on the last expression yields 27 summands; the map $(1_G \otimes \tau \otimes 1_S) \otimes 1_X$ transforms these to 27 summands in $X \otimes_A X \otimes_A X$. A quick check verifies that $1_X \otimes \lambda \otimes 1_X$ vanishes on all but two summands, namely

$$-\big((1_G \otimes 1_G) \otimes (1_S \otimes 1_S)\big) \otimes_A \big((1_G \otimes g^i \otimes 1_G) \otimes (1_S \otimes w \otimes 1_S)\big) \otimes_A \big((1_G \otimes 1_G) \otimes (1_S \otimes v \otimes 1_S)\big),$$
$$-\big((1_G \otimes 1_G) \otimes (1_S \otimes v \otimes 1_S)\big) \otimes_A \big(1_G \otimes g^i \otimes 1_G) \otimes (1_S \otimes w \otimes 1_S)\big) \otimes_A \big((1_G \otimes 1_G) \otimes (1_S \otimes 1_S)\big),$$

and we obtain

$$-\big((1_G \otimes 1_G) \otimes (1_S \otimes 1_S)\big) \otimes_A \big((ig^{i-1} \otimes 1_G) \otimes (1_S \otimes v \otimes 1_S)\big)$$
$$-\big((1_G \otimes 1_G) \otimes (1_S \otimes v \otimes 1_S)\big) \otimes_A \big((ig^{i-1} \otimes 1_G) \otimes (1_S \otimes 1_S)\big).$$

Applying ϕ_X followed by $f = \lambda$ or $f = \kappa$ gives 0 as w does not appear in the input.

Remark 5.1. The cocycles λ and κ above were not chosen randomly. These cocycles define a PBW deformation of $S \rtimes G$, and the zero brackets calculated above predict the PBW property. Indeed, in [8], we considered PBW deformations of $S \rtimes G$ given by analogs of Lusztig's graded Hecke algebras and symplectic reflection algebras over fields of positive characteristic. These algebras $\mathcal{H}_{\lambda,\kappa}$ depend on two parameters λ and κ with $\lambda : kG \otimes V \to kG$ and $\kappa : V \otimes V \to kG$. The Hochschild 2-cocycles above of the same name λ and κ are these parameters converted into cocycles on the resolution X; see [8, Example 2.2] and also [10, Section 5]. A necessary condition for the parameters λ and κ to define a PBW deformation is that

$$[\lambda, \lambda] = 0 \text{ and } [\lambda, \kappa] = 0$$

when the cochains κ and λ they define are cocycles. (More generally, we require that λ is a cocycle, $[\lambda, \lambda] = 0$, and $[\lambda, \lambda] = 2\partial^*\kappa$.) Thus knowing explicit values for brackets is helpful for finding new deformations. The cocycle δ above is included merely for illustration purposes; it provides an example of a nonzero Gerstenhaber bracket.

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