# Math 6810
# (Probability and Fractals)

## Spring 2016

## Lecture notes

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**Recommended reading:** (Do not purchase these books before consulting with your instructor!)

1. *Real Analysis* by H. L. Royden (4th edition), Prentice Hall.
2. *Probability and Measure* by P. Billingsley (3rd edition), Wiley.
3. *Probability with Martingales* by D. Williams, Cambridge University Press.
4. *Fractal Geometry: Foundations and Applications* by K. Falconer (2nd edition), Wiley.

# Chapter 5

# Brownian motion

## 5.1 Some general notions

By a stochastic process in continuous time we mean a collection $(X(t))_{t\in[0,\infty)}$ of random variables on a common probability space $(\Omega, \mathcal{F}, \mathrm{P})$, or sometimes a collection $(X(t))_{t\in[0,T]}$, where $T > 0$ is a constant. As in the discrete time case, we think of $t$ as a "time parameter". When it is necessary to indicate the dependence on $\omega$, we write $X(t, \omega)$.

By a *filtration* we mean here an increasing collection $(\mathcal{F}_t)_{t\geq 0}$ of sub-$\sigma$-algebras of $\mathcal{F}$; that is, $\mathcal{F}_s \subset \mathcal{F}_t$ for $s < t$.

**Definition 5.1.** A stochastic process $(X(t))_t$ is *adapted* to the filtration $(\mathcal{F}_t)_t$ if $X(t)$ is $\mathcal{F}_t$-measurable for each $t$.

**Definition 5.2.** A process $(X(t))_t$ is called a *submartingale* relative to the filtration $(\mathcal{F}_t)_t$ if:

(i) $(X(t))_t$ is adapted to $(\mathcal{F}_t)_t$;

(ii) $\mathrm{E}|X(t)| < \infty$ for all $t$; and

(iii) $\mathrm{E}[X(t)|\mathcal{F}_s] \geq X(s)$ a.s. for all $0 \leq s < t$.

A process $(X(t))_t$ is a *supermartingale* if $(-X(t))_t$ is a submartingale. A process that is both a submartingale and a supermartingale is called a *martingale.*

Some stochastic processes are constructed from the ground up; others are defined implicitly by a set of conditions. For this second type of process, it is necessary to prove that a stochastic process satisfying the conditions actually exists. Kolmogorov's existence theorem is an important tool for this.

**Definition 5.3.** Let $X = (X(t))_t$ be a stochastic process. The *finite-dimensional distributions* of $X$ are the probability measures

$$\mu_{t_1\ldots t_k}(B) := \mathrm{P}((X(t_1), \ldots, X(t_k)) \in B), \qquad k \in \mathbb{N}, \quad 0 \leq t_1 < t_2 < \ldots t_k, \quad B \in \mathcal{B}(\mathbb{R}^k).$$

Note that the family $\{\mu_{t_1\ldots t_k}\}$ satisfies the *consistency condition*

$$\begin{aligned}\mu_{t_1\ldots t_{i-1}t_{i+1}\ldots t_k}&(B_1 \times \cdots \times B_{i-1} \times B_{i+1} \times \cdots B_k) \\ &= \mu_{t_1\ldots t_k}(B_1 \times \cdots \times B_{i-1} \times \mathbb{R} \times B_{i+1} \times \cdots B_k),\end{aligned} \tag{5.1}$$

for all $i = 1, \ldots, k$, where $B_i \in \mathcal{B}(\mathbb{R})$ for each $i$.

**Theorem 5.4** (Kolmogorov's existence theorem). *If $\{\mu_{t_1 \ldots t_k}\}$ is a family of probability measures satisfying* (5.1)*, then there exists, on some probability space* $(\Omega, \mathcal{F}, \mathrm{P})$*, a stochastic process $(X(t))_t$ whose finite-dimensional distributions are $\mu_{t_1 \ldots t_k}$.*

*Proof.* See Billingsley, Section 36. □

## 5.2 Definition of Brownian motion

Botanist Robert Brown described the highly irregular motion of a pollen particle suspended in liquid in 1828. Albert Einstein gave a physical explanation for this motion and derived mathematical equations for it in 1905. The process that we now call *Brownian motion* was formulated rigorously (from a mathematical point of view) by Norbert Wiener in the 1920s, and because of his work, the process is also frequently called the *Wiener process.* In the 1940s, Paul Lévy analyzed Brownian motion more deeply and introduced the notion of local time, important for the theory of stochastic calculus. Around the same time, Kiyoshi Itô laid the groundwork for this new kind of calculus, publishing what is now known as Itô's rule, which replaces the chain rule from ordinary differential calculus. Brownian motion is now used in many areas, including physics, engineering and mathematical finance.

**Definition 5.5.** A (one-dimensional) *Brownian motion* is a stochastic process $\{W(t) : t \geq 0\} = \{W(t, \omega) : t \geq 0\}$ on some probability space $(\Omega, \mathcal{F}, \mathrm{P})$ with the following properties:

(i) $W(0) \equiv 0$.

(ii) (Independence of increments) For $0 \leq t_1 < t_2 < \cdots < t_n < \infty$, the increments $W(t_2) - W(t_1), W(t_3) - W(t_2), \ldots, W(t_n) - W(t_{n-1})$ are independent.

(iii) (Stationarity and normality of increments) For $0 < s < t$, the increment $W(t) - W(s)$ has a normal distribution with mean 0 and variance $t - s$.

(iv) (Continuity of sample paths) For each $\omega \in \Omega$, the function $t \mapsto W(t, \omega)$ is continuous.

The first question that needs to be addressed is whether such a process actually exists. It is not difficult to check that the stipulations (ii) and (iii) satisfy the consistency condition (5.1), so Kolmogorov's existence theorem implies the existence of a process $\{W(t)\}$ satisfying (i)-(iii). However, there is no guarantee that this process will have continuous sample paths. One way around this is to begin with a process $\{W(t)\}$ as given by Kolmogorov's theorem, and prove that the restriction of this process to dyadic rational time points is with probability one uniformly continuous on compact intervals. One can then redefine $W(t)$ at nondyadic $t$ by taking limits over dyadic rationals approaching $t$, which will guarantee continuity of $W(t)$. This approach, which can be found in Billingsley, section 37, is rather technical. Instead, we will construct Brownian motion from the ground up. But first, some preliminaries.

**Definition 5.6.** A process $\{X(t) : t \geq 0\}$ is *Gaussian* if every finite linear combination $a_1 X(t_1) + \cdots + a_n X(t_n)$ has a normal distribution, where $a_i \in \mathbb{R}$ and $t_i \geq 0$ for all $i$.

**Lemma 5.7.** *Brownian motion is a Gaussian process. It has mean and covariance functions given by*

$$\mu(t) := \mathrm{E}(W(t)) = 0,$$

*and*

$$r(s,t) := \mathrm{Cov}(W(s), W(t)) = \min\{s, t\}.$$

*Proof.* Let $0 \le t_1 < t_2 < \cdots < t_n$ and $a_1, \ldots, a_n \in \mathbb{R}$. Put $t_0 = 0$. Note that for each $i$, we can write $W(t_i)$ as a linear combination of the increments $W(t_1) - W(t_0), \ldots, W(t_n) - W(t_{n-1})$. But then $a_1 W(t_1) + \cdots + a_n W(t_n)$ is also a linear combination of these increments. Hence, $\{W(t)\}$ is Gaussian.

That the mean of $W(t)$ is zero is obvious. To compute the covariance, assume WLOG that $s < t$. Then $W(s)$ and $W(t) - W(s)$ are independent and have mean zero, so

$$\begin{aligned} \mathrm{E}[W(s)W(t)] &= \mathrm{E}\left[W(s)(W(t) - W(s))\right] + \mathrm{E}\left[(W(s))^2\right] \\ &= \mathrm{E}(W(s))\,\mathrm{E}(W(t) - W(s)) + \mathrm{Var}(W(s)) = s. \end{aligned}$$

Hence,

$$\mathrm{Cov}(W(s), W(t)) = \mathrm{E}(W(s)W(t)) - \mathrm{E}(W(s))\,\mathrm{E}(W(t)) = s,$$

as required. □

It can be shown that the finite-dimensional distributions of a Gaussian process are completely determined by its mean and covariance functions. Thus, if we can construct a Gaussian process with continuous sample paths and with the correct mean and covariance functions, then this process must be Brownian motion.

## 5.3 Construction of Brownian motion

To begin, note that it is sufficient to construct Brownian motion on $0 \le t \le 1$: we can then construct infinitely many independent copies of this Brownian motion and paste them together to obtain Brownian motion on $[0, \infty)$. Precisely, let $W_j(t)$, $j \in \mathbb{N}$ be independent Brownian motions on $[0, 1]$ and put

$$W(t) = \begin{cases} W_1(t), & 0 \le t < 1 \\ \sum_{j=1}^{n-1} W_j(1) + W_n(t-n), & n \le t < n+1, \quad n \in \mathbb{N}. \end{cases}$$

Then one checks easily that $W(t)$ is a Brownian motion on $[0, \infty)$.

**Step 1.** Define the *Haar functions*

$$H_1(t) = 1, \qquad 0 \le t \le 1$$

$$H_{2^n+1}(t) = \begin{cases} 2^{n/2}, & 0 \le t < 2^{-(n+1)} \\ -2^{n/2}, & 2^{-(n+1)} \le t \le 2^{-n} \\ 0, & \text{elsewhere} \end{cases} \qquad (n = 0, 1, 2, \ldots)$$

$$H_{2^n+j}(t) = H_{2^n+1}\left(t - \frac{j-1}{2^n}\right), \qquad j = 1, \ldots, 2^n, \qquad n \ge 0.$$

Define the *Schauder functions* by

$$S_k(t) = \int_0^t H_k(u)\,du, \qquad k \in \mathbb{N}, \quad 0 \le t \le 1.$$

Then $S_1(t) = t$, and for $n \ge 0$ and $1 \le j \le 2^n$, the graph of $S_{2^n+j}$ is a "tent" of height $2^{-(n+2)/2}$ over the interval $[(j-1)/2^n, j/2^n]$. Note that

$$S_{2^n+j}(t) S_{2^n+k}(t) = 0 \qquad \forall\, t \qquad \text{if } 1 \le k < j \le 2^n. \tag{5.2}$$

**Step 2.** Let $Z_k, k \in \mathbb{N}$ be independent standard normal r.v.'s. Put

$$W^{(n)}(t) = \sum_{k=1}^{2^n} Z_k S_k(t), \qquad 0 \le t \le 1, \quad n \ge 0.$$

**Lemma 5.8.** *As $n \to \infty$, $W^{(n)}$ converges uniformly on $[0,1]$ to a continuous function $W(t)$ with probability one.*

*Proof.* Let

$$M_n := \max_{1 \le j \le 2^{n-1}} |Z_{2^{n-1}+j}|, \qquad n \in \mathbb{N}.$$

For $x > 0$ and $k \in \mathbb{N}$, we have

$$\begin{aligned} \mathrm{P}(|Z_k| > x) &= 2\,\mathrm{P}(Z_k > x) = \sqrt{\frac{2}{\pi}} \int_x^\infty e^{-u^2/2}\,du \\ &\le \sqrt{\frac{2}{\pi}} \int_x^\infty \frac{u}{x} e^{-u^2/2}\,du = \sqrt{\frac{2}{\pi}} \frac{e^{-x^2/2}}{x}. \end{aligned}$$

Thus, for $n \in \mathbb{N}$,

$$\begin{aligned} \mathrm{P}(M_n > n) &= \mathrm{P}\left( \bigcup_{1 \le j \le 2^{n-1}} \{|Z_{2^{n-1}+j}| > n\} \right) \\ &\le \sum_{j=1}^{2^{n-1}} \mathrm{P}\left(|Z_{2^{n-1}+j}| > n\right) \\ &= 2^n\,\mathrm{P}(|Z_1| > n) \le \sqrt{\frac{2}{\pi}} \cdot \frac{2^n e^{-n^2/2}}{n}. \end{aligned}$$

Since

$$\sum_{n=1}^\infty \frac{2^n e^{-n^2/2}}{n} < \infty$$

(check!!), the first Borel-Cantelli lemma implies that

$$\mathrm{P}(M_n > n \text{ for infinitely many } n) = 0.$$

Hence, with probability one, there is an index $N$ such that $M_n \leq n$ whenever $n \geq N$. But then,

$$\sum_{k=2^N+1}^{\infty} |Z_k S_k(t)| \leq \sum_{n=N}^{\infty} M_{n+1} 2^{-(n+2)/2} \leq \sum_{n=N}^{\infty} (n+1) 2^{-(n+2)/2} < \infty$$

for all $0 \leq t \leq 1$. Thus, by the Cauchy criterion, $W^{(n)}(t)$ converges uniformly on $[0,1]$ to a function $W(t)$, which is continuous as the uniform limit of continous functions on $[0,1]$. □

**Step 3.** Note that $W^{(n)}$ is a mean-zero Gaussian process for each $n$. It can be shown (for instance using the method of characteristic functions) that the almost-sure limit of a sequence of Gaussian processes is Gaussian. Hence $W$ is Gaussian, and it remains to check that it has the correct mean and covariance functions.

**Exercise 5.9.** Show that

$$\sum_{k=1}^{\infty} S_k(t) < \infty \qquad \forall\, t \tag{5.3}$$

and

$$\sum_{j=1}^{\infty} \sum_{k=1}^{\infty} S_j(s) S_k(t) < \infty \qquad \forall\, s,t. \tag{5.4}$$

By (5.3) and Fubini's theorem,

$$\mathrm{E}\left( \sum_{k=1}^{\infty} |Z_k S_k(t)| \right) = \mathrm{E}\,|Z_1| \sum_{k=1}^{\infty} S_k(t) < \infty,$$

and so

$$\mathrm{E}(W(t)) = E\left( \sum_{k=1}^{\infty} Z_k S_k(t) \right) = \sum_{k=1}^{\infty} \mathrm{E}(Z_k) S_k(t) = 0.$$

To compute the covariance function, we need a little Fourier analysis. Verify that the functions $H_k$, $k \in \mathbb{N}$ form a complete orthonormal system; that is,

$$\int_0^1 H_j(t) H_k(t)\, dt = \begin{cases} 1, & j = k \\ 0, & j \neq k. \end{cases}$$

Therefore, Parseval's identity implies that for any two bounded functions $f, g$ on $[0,1]$,

$$\int_0^1 f(u) g(u)\, du = \sum_{k=1}^{\infty} a_k b_k,$$

where

$$a_k = \int_0^1 f(t) H_k(t)\, dt, \qquad b_k = \int_0^1 g(t) H_k(t)\, dt.$$

Apply this to $f = \chi_{[0,s]}$ and $g = \chi_{[0,t]}$. Then $a_k = S_k(s)$ and $b_k = S_k(t)$. Now

$$\begin{aligned}
\mathrm{E}\left(\sum_{j=1}^{\infty}\sum_{k=1}^{\infty}|Z_j Z_k S_j(s) S_k(t)|\right) &= \sum_{j=1}^{\infty}\sum_{k=1}^{\infty}\mathrm{E}\,|Z_j Z_k| S_j(s) S_k(t) \\
&\le \max\{\mathrm{E}\,|Z_1^2|, (\mathrm{E}\,|Z_1|)^2\}\sum_{j=1}^{\infty}\sum_{k=1}^{\infty} S_j(S) S_k(t) \\
&< \infty
\end{aligned}$$

by (5.4) and Fubini's theorem. Hence,

$$\begin{aligned}
r(s,t) = \mathrm{E}[W(s)W(t)] &= \sum_{j=1}^{\infty}\sum_{k=1}^{\infty}\mathrm{E}(Z_j Z_k) S_j(s) S_k(t) \\
&= \sum_{j=1}^{\infty}\mathrm{E}(Z_j^2) S_j(s) S_j(t) \qquad \text{(by independence of the } Z_j\text{)} \\
&= \sum_{j=1}^{\infty} S_j(s) S_j(t) \qquad \text{(since } \mathrm{E}(Z_j^2) = 1\text{)} \\
&= \int_0^1 \chi_{[0,s]}(u)\chi_{[0,t]}(u)\,du \qquad \text{(by Parseval's identity)} \\
&= \min\{s,t\}.
\end{aligned}$$

This shows that $\{W(t), 0 \le t \le 1\}$ is a Brownian motion.

## 5.4 Various properties of Brownian motion

**Exercise 5.10** (Brownian motion martingales)**.** Show that the following are martingales:

(i) $W(t)$

(ii) $W(t)^2 - t$

(iii) $e^{\lambda W(t) - \lambda^2 t/2}$, where $\lambda \in \mathbb{R}$.

**Proposition 5.11.** *On a set of probability one,*

$$\lim_{t\to\infty}\frac{W(t)}{t} = 0. \tag{5.5}$$

*Proof.* Writing $W(n) = W(0) + [W(1) - W(0)] + \cdots + [W(n) - W(n-1)]$, we see by the SLLN that

$$\lim_{n\to\infty}\frac{W(n)}{n} = 0 \qquad \text{a.s.}$$

It is now not hard to believe that (5.5) should hold, because for arbitrary $t > 0$, we can write

$$\frac{W(t)}{t} = \frac{[t]}{t}\cdot\left(\frac{W(t) - W([t])}{[t]} + \frac{W([t])}{[t]}\right),$$

where $[t]$ denotes the greatest integer in $t$. Intuitively, the first term in parentheses should approach zero as $t \to \infty$. But proving this requires some care; see Karatzas and Shreve, Problem 2.9.3. □

**Exercise 5.12.** Show that each of the following processes is a Brownian motion:

(i) (Reflection) $W_1(t) = -W(t)$

(ii) (Time scaling) $W_2(t) = cW(t/c^2)$, for fixed $c > 0$

(iii) (Time shift) $W_3(t) = W(t_0 + t) - W(t_0)$, for fixed $t_0 > 0$

(iv) (Time reversal) $W_4(t) = W(T - t) - W(T)$, $0 \le t \le T$, for fixed $T > 0$.

(v) (Time inversion)

$$W_5(t) = \begin{cases} tW(1/t), & t > 0 \\ 0, & t = 0 \end{cases}$$

(Hint: check that each process is a mean zero Gaussian process with continuous sample paths and the correct covariance function. For $W_5$, continuity at $t = 0$ follows from (5.5) and the substitution $u = 1/t$.)

Proposition 5.11 has the following strengthening, which specifies the maximum growth rate of Brownian paths. Its proof is beyond the scope of this course.

**Theorem 5.13** (Law of the iterated logarithm)**.** *We have*

$$\limsup_{t\to\infty} \frac{W(t)}{\sqrt{2t \log\log t}} = 1 \qquad a.s.$$

*and*

$$\liminf_{t\to\infty} \frac{W(t)}{\sqrt{2t \log\log t}} = -1 \qquad a.s.$$

(Check that these statements together imply (5.5)!)

**Exercise 5.14.** Use Theorem 5.13 to show that

$$\limsup_{t\downarrow 0} \frac{W(t)}{\sqrt{2t \log\log(1/t)}} = 1 \qquad \text{a.s.}$$

and the corresponding lim inf equals $-1$, almost surely.

We will use the result of the last exercise later to prove an important property of the zero set of Brownian motion.

**Theorem 5.15** (Nowhere differentiability of sample paths)**.** *For all $\omega$ outside a set of probability* 0, *$W(\cdot, \omega)$ is nowhere differentiable.*

Note that we avoid saying something like

$$P(\omega : W(\cdot, \omega) \text{ is nowhere differentiable}) = 1.$$

The reason is, that it is not at all clear whether the set in question is measurable. Proving the theorem entails finding a measurable subset of this set which has probability 1.

*Proof.* (We follow Billingsley, Theorem 37.3.) Put

$$\Delta_{n,k} = W\left(\frac{k+1}{2^n}\right) - W\left(\frac{k}{2^n}\right),$$

and let

$$X_{n,k} = \max\{|\Delta_{n,k}|, |\Delta_{n,k+1}|, |\Delta_{n,k+2}|\}.$$

Note that each $\Delta_{n,k}$ has the same distribution as $2^{-n/2}W(1)$, namely Normal$(0, 2^{-n})$. Furthermore, for fixed $n$, the $\Delta_{n,k}$ are independent. Thus, given $\varepsilon > 0$,

$$P(X_{n,k} \le \varepsilon) = \left[\, P(|W(1)| \le 2^{n/2}\varepsilon)\right]^3.$$

Now for $\alpha > 0$,

$$P(|W(1)| \le \alpha) = \int_{-\alpha}^{\alpha} \frac{1}{\sqrt{2\pi}} e^{-x^2/2}\, dx \le 2\alpha \frac{1}{\sqrt{2\pi}} < \alpha.$$

Hence,

$$P(X_{n,k} \le \varepsilon) \le (2^{n/2}\varepsilon)^3.$$

Put

$$Y_n = \min_{k \le n2^n} X_{n,k}.$$

Then

$$P(Y_n \le \varepsilon) \le n2^n (2^{n/2}\varepsilon)^3 = n2^{5n/2}\varepsilon^3. \tag{5.6}$$

We consider the upper and lower right-hand derivatives

$$D^+(t,\omega) = \limsup_{h \downarrow 0} \frac{W(t+h,\omega) - W(t,\omega)}{h},$$
$$D_+(t,\omega) = \liminf_{h \downarrow 0} \frac{W(t+h,\omega) - W(t,\omega)}{h}.$$

Define the set

$$E = \{\omega : \text{ there is } t \ge 0 \text{ such that } D^+(t,\omega) \text{ and } D_+(t,\omega) \text{ are both finite}\}.$$

Suppose $\omega \in E$; then we can find $t \ge 0$ and $K > 0$ (both depending on $\omega$) such that

$$\sup_{0 < h \le 1} \frac{|W(t+h,\omega) - W(t,\omega)|}{h} \le K. \tag{5.7}$$

Choose $n > \max\{2, 8K, t\}$. Let $k \in \mathbb{N}$ be such that $(k-1)/2^n \leq t < k/2^n$. Then $|t - i/2^n| \leq 1$ for $i = k, k+1, k+2, k+3$, and hence, by (5.7) and the triangle inequality,

$$X_{n,k}(\omega) \leq 2K(4/2^n) < n2^{-n}.$$

Since also $k - 1 \leq t2^n < n2^n$, it follows that $Y_n(\omega) \leq n2^{-n}$.

Define the set $A_n = \{Y_n \leq n2^{-n}\}$. The above argument shows that $E \subset \liminf A_n$. Note that each $A_n$ is measurable, and so $\liminf A_n$ is measurable. By (5.6),

$$\mathrm{P}(A_n) \leq n2^{5n/2}(n2^{-n})^3 = n^4 2^{-n/2} \to 0.$$

It follows (check!) that $\mathrm{P}(\liminf A_n) = 0$. And outside the set $\liminf A_n$, $W(\cdot, \omega)$ is nowhere differentiable (in fact, it does not have finite upper and lower right-hand derivatives anywhere). □

## 5.5 Stopping times and the strong Markov property

**Definition 5.16.** A *stopping time* relative to a filtration $(\mathcal{F}_t)_t$ is a $[0, \infty]$-valued r.v. $\tau$ such that for each $t > 0$,

$$\{\tau \leq t\} \in \mathcal{F}_t.$$

**Exercise 5.17.** Show that if $\tau$ is a stopping time, then for each $t > 0$,

$$\{\tau = t\} \in \mathcal{F}_t.$$

**Definition 5.18.** The *natural filtration* of a process $(X(t))_t$ is defined by $\mathcal{F}_t = \sigma(X(s) : 0 \leq s \leq t)$.

**Proposition 5.19.** *Let $(W(t))_t$ be a Brownian motion. Let $A \subset \mathbb{R}$ be a closed set, and define $\tau_A := \inf\{t > 0 : W(t) \in A\}$. Then $\tau_A$ is a stopping time relative to the natural filtration $(\mathcal{F}_t)_t$ of $(W(t))_t$.*

*Proof.* Observe, by continuity of sample paths, that

$$\{\tau_A \leq t\} = \bigcap_{n=1}^{\infty} \bigcup_{s \in [0,t] \cap \mathbb{Q}} \left\{ \operatorname{dist}(W(s), A) \leq \frac{1}{n} \right\},$$

and so $\{\tau_A \leq t\} \in \mathcal{F}_t$. □

**Definition 5.20.** For a stopping time $\tau$, define the collection

$$\mathcal{F}_\tau := \{A \in \mathcal{F} : A \cap \{\tau \leq t\} \in \mathcal{F}_t \ \forall t\}.$$

**Exercise 5.21.** Show that $\mathcal{F}_\tau$ is a $\sigma$-algebra and $\tau$ is $\mathcal{F}_\tau$-measurable.

From now on, let $(\mathcal{F}_t)_t$ denote the natural filtration of $(W(t))_t$. Fix $t_0 \geq 0$, and put

$$W'(t) = W(t_0 + t) - W(t_0), \qquad t \geq 0.$$

We have seen before that $W'(t)$ is a Brownian motion, and in view of the independent increments of $W(t)$, it is independent of $\mathcal{F}_t$. In particular, we have (check!)

$$\mathrm{P}(W(t_0+t) \le y|\mathcal{F}_{t_0}) = \mathrm{P}(W(t_0+t) \le y|W(t_0)) \qquad \text{a.s.} \tag{5.8}$$

We say that Brownian motion possesses the *Markov property.* More fully, we have, for $0 \le t_1 < t_2 < \cdots < t_k$ and $H \in \mathcal{B}(\mathbb{R}^k)$,

$$\begin{aligned}\mathrm{P}\big(\{(W'(t_1),\dots,W'(t_k)) \in H\} \cap A\big) &= \mathrm{P}\left((W'(t_1),\dots,W'(t_k)) \in H\right)\mathrm{P}(A)\\ &= \mathrm{P}\left((W(t_1),\dots,W(t_k)) \in H\right)\mathrm{P}(A), \qquad A \in \mathcal{F}_{t_0}.\end{aligned}$$

We now want to show that the above identities remain true when $t_0$ is replaced by a stopping time $\tau$. A process $X = (X(t))_t$ satisfying

$$\mathrm{P}(X(\tau+t) \le y|\mathcal{F}_\tau) = \mathrm{P}(X(\tau+t) \le y|X(\tau)) \qquad \text{a.s.} \tag{5.9}$$

for every finite stopping time $\tau$ is said to possess the *strong Markov property.* Note that the strong Markov property implies the Markov property, because any constant $t_0 > 0$ is a stopping time.

Some care is needed here; for instance, it is not a priory clear that $X(\tau+t)$ is even a random variable (i.e. is $\mathcal{F}$-measurable). We also must be careful with the use of conditional probabilities. We ignore these subtleties here; the details can be found for instance in Karatzas and Shreve, Chapter 2.

**Theorem 5.22** (Strong Markov property of Brownian motion)**.** *Let $\tau$ be a finite stopping time, and put*

$$W^*(t,\omega) := W(\tau(\omega)+t,\omega) - W(\tau(\omega),\omega), \qquad t \ge 0.$$

*Then $\{W^*(t) : t \ge 0\}$ is a Brownian motion independent of $\mathcal{F}_\tau$. That is, for $0 \le t_1 < t_2 < \cdots < t_k$, $H \in \mathcal{B}(\mathbb{R}^k)$ and $A \in \mathcal{F}_\tau$,*

$$\begin{aligned}\mathrm{P}\big(\{(W^*(t_1),\dots,W^*(t_k)) \in H\} \cap A\big) &= \mathrm{P}\left((W^*(t_1),\dots,W^*(t_k)) \in H\right)\mathrm{P}(A)\\ &= \mathrm{P}\left((W(t_1),\dots,W(t_k)) \in H\right)\mathrm{P}(A).\end{aligned} \tag{5.10}$$

*Proof.* See Billingsley, section 37. □

## 5.6 The reflection principle

The strong Markov property of Brownian motion says roughly speaking that at a finite stopping $\tau$, Brownian motion "starts afresh" from the (random) level $W(\tau)$. One consequence of this is the so-called reflection principle. Let $\tau$ be a finite stopping time, and define a new process

$$W'(t) = \begin{cases} W(t), & t \le \tau \\ W(\tau) - [W(t) - W(\tau)], & t \ge \tau. \end{cases} \tag{5.11}$$

Thus, the sample path of $W'(t)$ is the same as the sample path of $W(t)$ up to time $\tau$, and after that, it is the reflection of this path in the line $y = W(\tau)$.

**Theorem 5.23** (Reflection principle)**.** *The process* $\{W'(t) : t \geq 0\}$ *is a Brownian motion.*

This theorem is intuitively obvious from the strong Markov property: $W'(t)$ is a Brownian motion before time $\tau$, and it is a reflection of a Brownian motion, hence a Brownian motion, after time $\tau$, so it ought to be a Brownian motion "everywhere". However, since $\tau$ is random, some care is needed to check that the finite-dimensional distributions of $W'(t)$ are really the same as those of $W(t)$. (Continuity of sample paths is obvious from the definition of $W'(t)$.) See Billingsley, p. 512 for a precise proof.

We now define the *hitting time*

$$\tau_a := \inf\{t > 0 : W(t) \geq a\}, \qquad a > 0$$

and the maximum of Brownian motion up to time $t$,

$$M(t) := \max\{W(s) : 0 \leq s \leq t\}.$$

(By continuity of sample paths, this maximum is well defined.) Also by continuity of paths, $W(\tau_a) = a$ on the event $\{\tau_a < \infty\}$.

**Proposition 5.24.** *The stopping time* $\tau_a$ *is finite with probability one:* $\mathrm{P}(\tau_a < \infty) = 1$.

*Proof.* We use a martingale argument and the following fact: If $(M(t))$ is a martingale and $\tau$ a stopping time adapted to the natural filtration of $(M(t))$, then the *stopped process* $(M^\tau(t))$ defined by $M^\tau(t) := M(t \wedge \tau)$ (where $x \wedge y := \min\{x, y\}$) is a martingale. (This is easy to prove in the discrete time case. In the continuous time case, one approximates the stopping time $\tau$ by a stopping time taking only values in the set $\{k/2^n : k \in \mathbb{Z}, n \in \mathbb{N}\}$ and uses a limiting argument. Details can be found in Karatzas & Shreve.)

Recall from Exercise 5.10 that $X(t) := e^{uW(t)-u^2t/2}$ is a martingale for each $u > 0$, and note that $\mathrm{E}(X(t)) = 1$. Let $X^{\tau_a(}t)$ be the corresponding stopped martingale. Then

$$1 = \mathrm{E}(X^{\tau_a}(t)) = \mathrm{E}\left[\exp\left(uW(t \wedge \tau_a) - \frac{u^2(t \wedge \tau_a)}{2}\right)\right]. \tag{5.12}$$

Now observe that on $\{\tau_a \leq t\}$, $W(t \wedge \tau_a) = W(\tau_a) = a$, and on $\{\tau_a > t\}$, $W(t \wedge \tau_a) = W(t)$. Thus, the above equation becomes

$$1 = \mathrm{E}\left[\exp\left(ua - \frac{u^2\tau_a}{2}\right)\mathrm{I}(\tau_a \leq t)\right] + \mathrm{E}\left[\exp\left(uW(t) - \frac{u^2t}{2}\right)\mathrm{I}(\tau_a > t)\right]. \tag{5.13}$$

Now on $\{\tau_a > t\}$ we have $W(t) \leq a$, so the second term is bounded by $\exp(ua - u^2t/2)$, and hence tends to 0 as $t \to \infty$. On the other hand, $\{\tau_a \leq t\} \nearrow \{\tau_a < \infty\}$, so applying MCT to the first term gives

$$1 = \mathrm{E}\left[\exp\left(ua - \frac{u^2\tau_a}{2}\right)\mathrm{I}(\tau_a < \infty)\right], \tag{5.14}$$

which we can write as

$$\mathrm{E}\left[e^{-u^2\tau_a/2}\,\mathrm{I}(\tau_a < \infty)\right] = e^{-au}. \tag{5.15}$$

Using the convention $e^{-\infty} = 0$, we can freely add $\mathrm{E}[e^{-u^2\tau_a/2}\,\mathrm{I}(\tau_a = \infty)]$ to the left side and obtain

$$\mathrm{E}\left(e^{-u^2\tau_a/2}\right) = e^{-au}. \tag{5.16}$$

Now substitute $\lambda = u^2/2$, so $u = \sqrt{2\lambda}$; then the last equation becomes

$$\psi(\lambda) := \mathrm{E}\left(e^{-\lambda\tau_a}\right) = e^{-a\sqrt{2\lambda}}, \qquad \text{for all } \lambda > 0. \tag{5.17}$$

We call $\psi(\lambda)$ the *Laplace transform* of $\tau_a$. Finally, check that

$$\mathrm{P}(\tau_a < \infty) = \lim_{\lambda\downarrow 0} \psi(\lambda) = 1,$$

completing the proof. □

Observe that we could also have concluded the finiteness of $\tau_a$ from the law of the iterated logarithm (LIL); see Theorem 5.13. However, the LIL is a deep theorem whose proof is heavy on analysis. The above approach is more elementary, and, most importantly, very probabilistic, and a nice illustration of the use of martingales in probability.

We will use the reflection principle to derive the distributions of $\tau_a$ and $M(t)$. Define $W'(t)$ by (5.11) with $\tau = \tau_a$. Then

$$\begin{aligned}
\mathrm{P}(M(t) > a) &= \mathrm{P}(M(t) > a, W(t) > a) + \mathrm{P}(M(t) > a, W(t) \le a)\\
&= \mathrm{P}(W(t) > a) + \mathrm{P}(W'(t) \ge a)\\
&= \mathrm{P}(W(t) > a) + \mathrm{P}(W(t) \ge a)\\
&= 2\,\mathrm{P}(W(t) > a).
\end{aligned}$$

Here the second equality follows by definition of $W'(t)$, the third by Theorem 5.23, and the last since $W(t)$ is continuous. We conclude that

$$\mathrm{P}(M(t) > a) = \mathrm{P}(|W(t)| > a), \qquad a > 0; \tag{5.18}$$

in other words, $M(t) \stackrel{d}{=} |W(t)|$ for every $t \ge 0$. (Be careful, however: the *processes* $(M(t))_t$ and $(|W(t)|)_t$ do not have the same law; in particular, $M(t)$ is increasing in $t$, whereas $|W(t)|$ almost surely is not.)

**Exercise 5.25.** Derive from (5.18) the density of $M(t)$, and using the relation $\tau_a \le t \Leftrightarrow M(t) \ge a$ find the density of $\tau_a$.

## 5.7 The graph of Brownian motion

Our goal in this section is to compute the Hausdorff dimension of the graph $\mathrm{Graph}(W) = \{(t, W(t)) : 0 \le t \le 1\}$. We follow Falconer, chapter 16.

**Exercise 5.26.** Show that for any $\alpha > 0$ and $\beta > 0$,

$$\lim_{h\downarrow 0} \frac{\exp(-h^{-\alpha})}{h^\beta} = 0.$$

(It helps to make the substitution $x = 1/h$.)

**Theorem 5.27.** *For each $0 < \alpha < 1/2$, Brownian motion is Hölder continuous with exponent $\alpha$; precisely, there is with probability* 1 *a constant $b$ (depending only on $\alpha$) and a (random) number $h_0 > 0$ such that*

$$|W(t+h) - W(t)| \le b|h|^\alpha, \qquad \text{for all } t \in [0,1] \text{ and } 0 < |h| < h_0. \tag{5.19}$$

*Proof.* Let $Z$ be a standard normal random variable. For fixed $t$ and $h > 0$, we have by the scaling property,

$$\begin{aligned} \mathrm{P}(|W(t+h) - W(t)| > h^\alpha) &= \mathrm{P}(|Z| > h^{\alpha-1/2}) \\ &= 2\int_{h^{\alpha-1/2}}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-u^2/2} du \\ &\le c_1 \int_{h^{\alpha-1/2}}^{\infty} e^{-u} du \\ &= c_1 \exp(-h^{\alpha-1/2}) \\ &\le c_2 h^2, \end{aligned} \tag{5.20}$$

where $c_1$ and $c_2$ are constants that do not depend on $t$ or $h$, and in the last step we used Exercise 5.26. Now let

$$A_n := \left\{ \left| W\left(\frac{j}{2^n}\right) - W\left(\frac{j-1}{2^n}\right) \right| > 2^{-\alpha n} \text{ for some } 1 \le j \le 2^n \right\}.$$

We see from (5.20) that

$$\mathrm{P}(A_n) \le c_2 2^n 2^{-2n} = c_2 2^{-n}. \tag{5.21}$$

Thus, by the Borel-Cantelli lemma, $\mathrm{P}(\limsup A_n) = 0$, so with probability 1 there is an index $N$ such that

$$\left| W\left(\frac{j}{2^n}\right) - W\left(\frac{j-1}{2^n}\right) \right| \le 2^{-\alpha n} \qquad \text{for all } n > N \text{ and } 1 \le j \le 2^n. \tag{5.22}$$

Set $h_0 = 2^{-N}$. If $0 < h < h_0$, the interval $[t, t+h]$ can be written (except possibly for its endpoints) as a countable union of contiguous intervals $[(j-1)/2^n, j/2^n]$ with $2^{-n} \le h$ and $1 \le j \le 2^n$, and with no more than two intervals of any one length. By continuity of $W(t)$ and repeated application of the triangle inequality, it follows by (5.22) that if $k$ is the smallest integer with $2^{-k} \le h$,

$$|W(t+h) - W(t)| \le 2\sum_{n=k}^{\infty} 2^{-\alpha n} = 2\frac{2^{-\alpha k}}{1 - 2^{-\alpha}} \le \frac{2h^\alpha}{1 - 2^{-\alpha}}$$

for all $t \in [0,1]$, with probability 1. □

**Remark 5.28.** It is possible to prove a slighly stronger result, namely that with probability 1 there is a constant $b$ such that

$$|W(t+h) - W(t)| \le b\sqrt{h \log_2\left(\frac{1}{h}\right)} \qquad \text{for all } 0 \le t \le 1 \text{ and } h > 0.$$

**Theorem 5.29.** *With probability 1,* $\dim_H \text{Graph}(W) = \dim_B \text{Graph}(W) = 3/2$.

*Proof.* From Theorem 5.27 and Proposition 3.41(i), it follows that $\overline{\dim}_B \text{ Graph}(W) \leq 2-\alpha$ a.s. for all $0 < \alpha < 1/2$, and hence, $\overline{\dim}_B \text{ Graph}(W) \leq 3/2$ a.s.

For the lower bound, we use the potential theoretic method. Let $c = \sqrt{2/\pi}$. Then for $s > 1$,

$$\begin{aligned} \mathrm{E}\left[\left((W(t+h) - W(t))^2 + h^2\right)^{-s/2}\right] &= \int_{-\infty}^{\infty} (x^2 + h^2)^{-s/2} \frac{e^{-x^2/2h}}{\sqrt{2\pi h}} dx \\ &= ch^{-1/2} \int_0^{\infty} (x^2 + h^2)^{-s/2} e^{-x^2/2h} dx \\ &= \frac{c}{2} \int_0^{\infty} (wh + h^2)^{-s/2} w^{-1/2} e^{-w/2} dw \\ &\leq \frac{c}{2} \left( \int_0^h (h^2)^{-s/2} w^{-1/2} dw + \int_h^{\infty} (wh)^{-s/2} w^{-1/2} dw \right) \\ &= c_1(s) h^{1/2-s}, \end{aligned}$$

where we used the substitution $w = x^2/h$ and the last step follows by direct calculation (do it!). Now we define a random measure $\mu_W$ by lifting Lebesgue measure from the $t$-axis to the graph of $W$:

$$\mu_W(A) := \mathcal{L}(\{t \in [0,1] : (t, W(t)) \in A\}), \qquad A \in \mathcal{B}(\mathbb{R}^2),$$

just as we did in the proof of Theorem 4.8. For $1 < s < 3/2$, we now obtain from the above estimate, using the Pythagorean theorem:

$$\begin{aligned} \mathrm{E}\big(I_s(\mu_W)\big) &= \mathrm{E}\left( \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} |\mathbf{x} - \mathbf{y}|^{-s} d\mu_W(\mathbf{x}) d\mu_W(\mathbf{y}) \right) \\ &= \int_0^1 \int_0^1 \mathrm{E}\left[\left((W(t) - W(u))^2 + (t-u)^2\right)^{-s/2}\right] dt du \\ &\leq c_1(s) \int_0^1 \int_0^1 |t-u|^{1/2-s} dt du < \infty. \end{aligned}$$

Thus, $I_s(\mu_W) < \infty$ a.s., and by Theorem 4.3, $\dim_H \text{Graph}(W) \geq s$ a.s., for all $1 < s < 3/2$. But then $\dim_H \text{Graph}(W) \geq 3/2$ a.s., completing the proof. □

## 5.8 Multidimensional Brownian motion

If we are given $n$ independent Brownian motions $(W_1(t))_t, \ldots, (W_n(t))_t$, we can put $W(t) := (W_1(t), \ldots, W_n(t))$. We call the process $(W(t))_t$ a *Brownian motion in* $\mathbb{R}^n$, or $n$-dimensional Brownian motion. We will be interested in the *Brownian trail* $\{W(t) : t \geq 0\}$. Note that when $n = 1$, the Brownian trail is an interval (in fact, with probability one, the whole real line), by continuity of paths, and so it has positive Lebesgue measure and Hausdorff dimension 1. For $n \geq 2$, however, the situation is more interesting:

**Theorem 5.30.** *For all* $n \geq 2$*, the Brownian trail in* $\mathbb{R}^n$ *has Hausdorff and box-counting dimension 2 with probability 1.*

We first need a couple of lemmas.

**Lemma 5.31.** *Let $Z_1, \dots, Z_n$ be independent standard normal random variables, and $Z := (Z_1, \dots, Z_n)$. Then $|Z|$ has density*

$$f_{|Z|}(r) := c_n r^{n-1} e^{-r^2/2}, \qquad r > 0,$$

*where $c_n = (2\pi)^{-n/2} a_n$, with $a_n$ being the $n-1$ dimensional area of the unit sphere in $\mathbb{R}^n$.*

*Proof.* Let $E$ initially be a rectangle of the form $E = [a_1, b_1] \times \cdots \times [a_n, b_n]$. Then

$$\mathrm{P}(Z \in E) = \int_{a_1}^{b_1} \cdots \int_{a_n}^{b_n} \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi}} e^{-z_i^2/2} dz_n \cdots dz_1 = (2\pi)^{-n/2} \int_E e^{-|z|^2/2} dz.$$

Since the collection of rectangles of this form is a semi-ring that generates the Borel $\sigma$-algebra in $\mathbb{R}^n$, it follows from the uniqueness part of Carathéodory's theorem that

$$\mathrm{P}(Z \in E) = (2\pi)^{-n/2} \int_E e^{-|z|^2/2} dz \qquad \text{for all } E \in \mathcal{B}(\mathbb{R}^n).$$

Take now $E = B(0, \rho)$ where $\rho > 0$. Then we obtain, by converting to spherical coordinates,

$$\mathrm{P}(|Z| \le \rho) = (2\pi)^{-n/2} \int_{B(0,\rho)} e^{-|z|^2/2} dz = (2\pi)^{-n/2} a_n \int_0^\rho r^{n-1} e^{-r^2/2} dr.$$

The lemma now follows by differentiating. □

**Lemma 5.32.** *Let $F \subset \mathbb{R}^n$ and suppose $f : F \to \mathbb{R}^n$ satisfies the Hölder condition*

$$|f(x) - f(y)| \le c|x - y|^\alpha, \qquad \textit{for all } x, y \in F,$$

*where $\alpha > 0$ and $c > 0$. Then $\dim_H f(F) \le (1/\alpha) \dim_H F$.*

*Proof.* Fix $s > \dim_H F$. If $\{U_i\}$ is a $\delta$-cover of $F$, then $|f(F \cap U_i)| \le c|U_i|^\alpha$, so $\{f(F \cap U_i)\}$ is an $\varepsilon$-cover of $f(F)$, where $\varepsilon = c\delta^\alpha$. Hence,

$$\mathcal{H}_\varepsilon^{s/\alpha}(f(F)) \le \mathcal{H}_\delta^s(F).$$

Letting $\delta \downarrow 0$ gives $\mathcal{H}^{s/\alpha}(f(F)) = 0$, and so $\dim_H f(F) \le s/\alpha$. □

*Proof of Theorem 5.30.* Let $0 < \alpha < 1/2$. By Theorem 5.27, each component $W_i$ is Hölder continuous with exponent $\alpha$. Thus (check!) the random function $W : [0, 1] \to \mathbb{R}^n$ satisfies with probability one the condition of Lemma 5.32, so $\dim_H W([0, 1]) \le (1/\alpha) \dim_H [0, 1] = 1/\alpha$, and a similar inequality holds for the box-counting dimension. Letting $\alpha$ increase to $1/2$ along a sequence $\{\alpha_n\}$ gives the upper bound.

For the lower bound we use the potential-theoretic method. We first recall that for $t \in [0, 1]$ and $h > 0$, $W_i(t + h) - W_i(t) \sim \text{Normal}(0, h) \sim \sqrt{h} Z_i$ for $i = 1, \dots, n$, where $Z_i$ is standard normal, and then $|W(t + h) - W(t)| \sim \sqrt{h}|Z|$, with $Z$ as in Lemma 5.31. By that lemma and a change of variable, $|W(t + h) - W(t)|$ thus has density

$$\frac{1}{\sqrt{h}} f_{|Z|}(r/\sqrt{h}) = c_n h^{-n/2} r^{n-1} e^{-r^2/2h}.$$

It follows that, for $1 < s < 2$,

$$\begin{aligned}\mathrm{E}\left(|W(t+h)-W(t)|^{-s}\right) &= c_n h^{-n/2}\int_0^\infty r^{-s}r^{n-1}e^{-r^2/2h}dr \\ &= \frac{1}{2}c_n h^{-s/2}\int_0^\infty w^{(n-s-2)/2}e^{-w/2}dw \\ &= \tilde{c}h^{-s/2},\end{aligned}$$

where $\tilde{c}$ does not depend on $h$ or $t$. (Note that the integral converges since $(n-s-2)/2 \geq -s/2 > -1$.)

Now define a (random) mass distribution $\mu_W$ on the trail $W([0,1])$ by

$$\mu_W(A) := \mathcal{L}(\{t \in [0,1] : W(t) \in A\}), \qquad A \in \mathcal{B}(\mathbb{R}^n).$$

Then for any function $g$, $\int_{\mathbb{R}^n} g(\mathbf{x})d\mu_W(\mathbf{x}) = \int_0^1 g(W(t))dt$, so

$$\begin{aligned}\mathrm{E}(I_s(\mu_W)) &= \mathrm{E}\left(\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\frac{d\mu_W(\mathbf{x})d\mu_W(\mathbf{y})}{|\mathbf{x}-\mathbf{y}|^s}\right) \\ &= \mathrm{E}\left(\int_0^1\int_0^1 \frac{dtdu}{|W(t)-W(u)|^s}\right) \\ &= \int_0^1\int_0^1 \mathrm{E}\left(|W(t)-W(u)|^{-s}\right)dtdu \\ &= \tilde{c}\int_0^1\int_0^1 |t-u|^{-s/2}dtdu \\ &< \infty,\end{aligned}$$

for all $1 < s < 2$. Thus, as before, $\dim_H W([0,1]) \geq 2$ with probability 1. □

**Remark 5.33.** It can be shown that $\mathcal{H}^2(W([0,1])) = 0$ for Brownian trail in $\mathbb{R}^n$ with $n \geq 2$. In particular, Brownian trail has Lebesgue measure zero, even in $\mathbb{R}^2$.

## 5.9 The zero set of Brownian motion

From now on, let $W(t)$ once again denote a one-dimensional Brownian motion. Our goal is to study the zero set

$$\mathcal{Z}_W := \{t \geq 0 : W(t) = 0\}.$$

We follow lecture notes of Y. Peres (see www.stat.berkeley.edu/∼peres/bmall.pdf).

The Law of the Iterated Logarithm in the form of Exercise 5.14 has the remarkable consequence that, on any time interval $[0,\varepsilon]$, $W(t)$ changes sign infinitely many times with probability one. Since $W(t)$ is continuous in $t$, this means that $W(t) = 0$ for infinitely many $t$ in $[0,\varepsilon]$ with probability one; in other words, 0 is an accumulation point of $\mathcal{Z}_W$. In fact, more can be proved:

**Theorem 5.34** (Zero set of Brownian motion)**.** *The set $\mathcal{Z}_W$ is with probability one an uncountable closed set without isolated points.*

*Proof.* By continuity of $W(t)$, it is clear that $\mathcal{Z}_W$ is closed. Since 0 is an accumulation point of $\mathcal{Z}_W$ from the right, it follows from the strong Markov property that any stopping time $\tau$ with $\mathrm{P}(W(\tau) = 0) = 1$ is an accumulation point from the right. This holds in particular for stopping times of the form $\tau_q := \inf\{t > q : W(t) = 0\}$, where $q \in [0, \infty) \cap \mathbb{Q}$. Since there are only countably many such $q$, we conclude that with probability one, $\tau_q$ is an accumulation point of $\mathcal{Z}_W$ from the right for each rational $q$. Let $t > 0$ and suppose $t$ is a point of $\mathcal{Z}_W$ isolated from the right. Then we can take a sequence $(q_n)$ of rationals such that $q_n \nearrow t$, and define the stopping times $\tau_n := \tau_{q_n}$. Since $W(t) = 0$ and $\tau_n$ is an accumulation point from the right but $t$ is not, $q_n \leq \tau_n < t$ for each $n$. But $q_n \to t$, so $t$ is an accumulation point of $\mathcal{Z}_W$ from the left. Thus, $\mathcal{Z}_W$ is closed without isolated points. Moreover, $\mathcal{Z}_W$ is nonempty (since it contains 0). In other words, $\mathcal{Z}_W$ is a perfect set, and it well known (and a nice exercise!) that any perfect set is uncountable. □

The above theorem says that $\mathcal{Z}_W$ is essentially a random Cantor set, so it is interesting to ask about its Hausdorff dimension. Calculating this is the end goal of the remainder of this section. For the lower bound, we shall use the *maximum process*

$$M(t) := \max\{W(s) : 0 \leq s \leq t\}, \qquad t \geq 0,$$

and the *reflection of Brownian motion in its maximum*,

$$Y(t) := M(t) - W(t), \qquad t \geq 0.$$

We will show that, *as a process*, $(Y(t))_t$ has the same distribution as $(|W(t)|)_t$ (Brownian motion reflected in the origin), and so the zero sets of these processes have almost surely the same Hausdorff dimension. It turns out that a lower bound for the dimension of the zero set

$$\mathcal{Z}_Y := \{t \geq 0 : Y(t) = 0\}$$

is much easier to come by:

**Proposition 5.35.** *With probability one,* $\dim_H \mathcal{Z}_Y \geq 1/2$.

*Proof.* Since $M(t)$ is nondecreasing, it induces a unique random Borel measure $\mu$ satisfying $\mu((s,t]) = M(t) - M(s)$, for all $0 \leq s < t < \infty$. This measure $\mu$ is supported on $\mathcal{Z}_Y$, because $M(t)$ can increase only when the Brownian motion is at its maximum, in which case $Y(t) = 0$. So $\mu$ is a mass distribution on $\mathcal{Z}_Y$. By Theorem 5.27, there is for every $0 < \alpha < 1/2$ a random constant $C_\alpha$ such that

$$\mu((s,t]) = M(t) - M(s) \leq \max_{0 \leq h \leq t-s} W(s+h) - W(s) \leq C_\alpha (t-s)^\alpha.$$

Hence, by the mass distribution principle, $\dim_H \mathcal{Z}_Y \geq \alpha$ with probability 1. Taking a sequence $\{\alpha_n\}$ with $\alpha_n \nearrow 1/2$ completes the proof. □

### 5.9.1 General Markov processes

We now need to make precise that the processes $(Y(t))_t$ and $(|W(t)|)_t$ have the same distribution, or law. First, we can consider any stochastic process $(X(t) : t \geq 0)$ as a random element of $\mathbb{R}^{[0,\infty)}$, the set of all function from $[0,\infty)$ to $\mathbb{R}$. We equip $\mathbb{R}^{[0,\infty)}$ with the $\sigma$-algebra $\mathcal{F}_C$ generated by the *cylinder sets*

$$\{x \in \mathbb{R}^{[0,\infty)} : (x(t_1), x(t_2), \dots, x(t_n)) \in A\}, \qquad n \in \mathbb{N}, \quad t_1, t_2, \dots, t_n \geq 0, \quad A \in \mathcal{B}(\mathbb{R}^n).$$

**Definition 5.36.** Two processes $(X(t) : t \geq 0)$ and $(X'(t) : t \geq 0)$ are *equal in law* (or *in distribution*), denoted $(X(t))_t \stackrel{d}{=} (X'(t))_t$, if $\mathrm{P}(X(\cdot) \in E) = \mathrm{P}(X'(\cdot) \in E)$ for every $E \in \mathcal{F}_C$.

If two processes are equal in law, their sample paths have the same almost-sure properties. We will use this fact below.

**Definition 5.37.** A *Markov transition kernel* is a function $p : [0,\infty) \times \mathbb{R} \times \mathcal{B}(\mathbb{R}) \to [0,1]$ satisfying:

(i) $p(t, x, \cdot)$ is a Borel probability measure on $\mathbb{R}$ for all $t \geq 0$ and $x \in \mathbb{R}$;

(ii) $p(\cdot, \cdot, A)$ is Borel measurable in $(t, x)$ for every $A \in \mathcal{B}(\mathbb{R})$;

(iii) For all $t, s > 0$, $x \in \mathbb{R}$ and $A \in \mathcal{B}(\mathbb{R})$,

$$p(t+s, x, A) = \int_{\mathbb{R}} p(t, y, A) p(s, x, dy).$$

**Definition 5.38.** A process $(X(t) : t \geq 0)$ is a (time-homogeneous) *Markov process* with transition kernel $p(t, x, A)$ if for all $t > s$ and $A \in \mathcal{B}(\mathbb{R})$ we have

$$\mathrm{P}(X(t) \in A | \mathcal{F}_s) = p(t-s, X(s), A),$$

where $\mathcal{F}_s = \sigma\{X(u) : 0 \leq u \leq s\}$.

**Example 5.39.** Brownian motion is a Markov process with transition kernel

$$p(t, x, A) = \int_A \frac{1}{\sqrt{2\pi t}} e^{-(y-x)^2/2t} dy.$$

That is, $p(t, x, \cdot)$ is the $N(x, t)$ (Normal$(x, t)$) distribution for all $(t, x)$.

**Example 5.40.** Reflected Brownian motion $|W(t)|$ is Markov with transition kernel $p(t, x, \cdot)$ being the distribution of $|Z|$, where $Z \sim N(x, t)$.

**Example 5.41.** The maximum process $M(t)$ is *not* Markov. Why not?

**Theorem 5.42.** *The process $Y(t) = M(t) - W(t)$ is Markov, and its transition kernel is the distribution of $|Z|$, where $Z \sim N(x, t)$.*

*Proof.* Fix $s \geq 0$, and define the processes $\hat{W}(t) := W(s+t) - W(s)$ and $\hat{M}(t) := \max\{\hat{W}(u) : 0 \leq u \leq t\}$. Recall from Exercise 5.12 that $\hat{W}(t)$ is again a Brownian motion, so $(\hat{W}(t), \hat{M}(t))_t \stackrel{d}{=} (W(t), M(t))_t$. Let $\mathcal{F}(s) := \sigma\{W(u) : 0 \leq u \leq s\}$. We must show that conditional on $\mathcal{F}(s)$ and $Y(s) = y$, $Y(s+t) \stackrel{d}{=} |y + \hat{W}(t)|$.

Write $x \vee y := \max\{x, y\}$, and note that $x \vee y - z = (x - z) \vee (y - z)$. Thus, we can write

$$\begin{aligned} Y(s+t) &= M(s+t) - W(s+t) \\ &= M(s) \vee \big(W(s) + \hat{M}(t)\big) - \big(W(s) + \hat{W}(t)\big) \\ &= Y(s) \vee \hat{M}(t) - \hat{W}(t). \end{aligned}$$

As a result, we need to check that $y \vee \hat{M}(t) - \hat{W}(t) \stackrel{d}{=} |y + \hat{W}(t)|$, or equivalently, that

$$y \vee M(t) - W(t) \stackrel{d}{=} |y - W(t)|, \tag{5.23}$$

since $W(t) \stackrel{d}{=} -W(t)$. For $a \geq 0$, write

$$\mathrm{P}(y \vee M(t) - W(t) > a) = \mathrm{P}(y - W(t) > a) + \mathrm{P}(y - W(t) \leq a, M(t) - W(t) > a). \tag{5.24}$$

To study the second term we define $B(u) := W(t-u) - W(t)$, and note by Exercise 5.12 that $(B(u) : 0 \leq u \leq t)$ is again a Brownian motion. Let $M_B(t) := \max\{B(u) : 0 \leq u \leq t\}$. Then $M_B(t) = M(t) - W(t)$, and since $B(t) = -W(t)$, we have

$$\mathrm{P}(y - W(t) \leq a, M(t) - W(t) > a) = \mathrm{P}(y + B(t) \leq a, M_B(t) > a).$$

Now let $B'(t)$ be the process obtained by reflecting $B(u)$ at the first time it hits $a$; that is, $B'(u)$ is defined as in (5.11) with $\tau = \tau_a := \inf\{u > 0 : B(u) = a\}$. By the reflection principle, $B'(t)$ is also a Brownian motion, and

$$\mathrm{P}(y + B(t) \leq a, M_B(t) > a) = \mathrm{P}(B'(t) \geq a + y) = \mathrm{P}(W(t) \geq a + y).$$

Finally, adding the two terms in (5.24) together, we obtain

$$\mathrm{P}(y \vee M(t) - W(t) > a) = \mathrm{P}(y - W(t) > a) + \mathrm{P}(W(t) - y \geq a) = \mathrm{P}(|y - W(t)| > a),$$

proving (5.23), as required. □

**Proposition 5.43.** *Two Markov processes with continuous paths, with the same initial distribution and the same transition kernel, are identical in law.*

(The proof of this proposition is beyond the scope of this course.)

**Corollary 5.44.** *The processes $(Y(t))_t$ and $(|W(t)|)_t$ are equal in law.*

As a result, the zero sets $\mathcal{Z}_W$ and $\mathcal{Z}_Y$ have the same distribution, and so $\dim_H \mathcal{Z}_W \stackrel{d}{=} \dim_H \mathcal{Z}_Y$. Thus, by Proposition 5.35, $\dim_H \mathcal{Z}_W \geq 1/2$ a.s.

We will now show that almost surely, $\overline{\dim}_B\, (\mathcal{Z}_W \cap [0,1] \leq 1/2$. We begin with a lemma.

**Lemma 5.45.** *For $a > 0$ and $\delta > 0$, we have*

$$\mathrm{P}(W(t) = 0 \textit{ for some } t \in [a, a+\delta]) = \frac{2}{\pi} \arctan \sqrt{\frac{\delta}{a}}.$$

*Proof.* First, for $x > 0$, we have by the the Markov property and symmetry of Brownian motion and the reflection principle,

$$\begin{aligned}
\mathrm{P}(\exists\, t \in [a, a+\delta] : W(t) = 0 | W(a) = x) &= \mathrm{P}\left(\min_{a \le t \le a+\delta} W(t) < 0 \big| W(a) = x\right) \\
&= \mathrm{P}\left(M(\delta) > x\right) = 2\,\mathrm{P}(W(\delta) > x).
\end{aligned}$$

Similarly, for $x < 0$, we get $\mathrm{P}(\exists\, t \in [a, a+\delta] : W(t) = 0 | W(a) = x) = 2\,\mathrm{P}(W(\delta) > |x|)$. Thus,

$$\begin{aligned}
\mathrm{P}(\exists\, t \in [a, a+\delta] : W(t) = 0) &= \int_{-\infty}^{\infty} 2\,\mathrm{P}(W(\delta) > |x|)\,\mathrm{P}(W(a) \in dx) \\
&= 4 \int_0^{\infty} \int_x^{\infty} \frac{1}{\sqrt{2\pi a \delta}} \exp\left(-\frac{y^2}{2\delta} - \frac{x^2}{2a}\right) \\
&= \frac{2}{\pi} \arctan \sqrt{\frac{\delta}{a}}.
\end{aligned}$$

It is left as a calculus exercise to verify the last step above! (Hint: make the substitution $w = y\sqrt{a/\delta}$, then transform to polar coordinates.) □

As a consequence (and this is all we'll need),

$$\mathrm{P}(W(t) = 0 \text{ for some } t \in [a, a+\delta]) < \sqrt{\frac{\delta}{a}} \qquad \text{for all } a > 0,\ \delta > 0. \tag{5.25}$$

**Proposition 5.46.** *With probablity one,* $\overline{\dim}_B\left(\mathcal{Z}_W \cap [0,1]\right) \le 1/2$.

*Proof.* Let $N_m$ be the number of intervals $[(k-1)/2^m, k/2^m]$, $k = 1, 2, \ldots, 2^m$ that intersect $\mathcal{Z}_W$. Taking $a = (k-1)/2^m$ and $\delta = 1/2^m$ in (5.25), we obtain

$$\begin{aligned}
\mathrm{E}(N_m) &= \sum_{k=1}^{2^m} \mathrm{P}(\exists\, t \in [(k-1)/2^m, k/2^m] : W(t) = 0) \\
&\le 1 + \sum_{k=2}^{2^m} \frac{1}{\sqrt{k-1}} < 2 + \int_1^{2^m} \frac{dx}{\sqrt{x}} \\
&< 2(1 + 2^{m/2}) \le 3 \cdot 2^{m/2},
\end{aligned}$$

for all $m \ge 2$. As a result, we have for each $\varepsilon > 0$,

$$\mathrm{P}\left(N_m > 2^{m(1/2+\varepsilon)}\right) \le \frac{\mathrm{E}(N_m)}{2^{m(1/2+\varepsilon)}} \le \frac{3 \cdot 2^{m/2}}{2^{m(1/2+\varepsilon)}} = \frac{3}{2^{m\varepsilon}},$$

so by the Borel-Cantelli lemma, $N_m \le 2^{m(1/2+\varepsilon)}$ for all sufficiently large $m$, with probability 1. Hence, $\overline{\dim}_B\left(\mathcal{Z}_W \cap [0,1]\right) \le 1/2 + \varepsilon$ a.s., and letting $\varepsilon \downarrow 0$ along a sequence $(\varepsilon_n)$ the proposition follows. □

Combining the results of this section, we finally obtain:

**Theorem 5.47.** *We have* $\dim_H \mathcal{Z}_W = 1/2$ *and* $\dim_H(\mathcal{Z}_W \cap [0,1]) = \dim_B(\mathcal{Z}_W \cap [0,1]) = 1/2$ *almost surely.*

**Corollary 5.48.** *For every* $a \in \mathbb{R}$*, the set* $\{t \geq 0 : W(t) = a\}$ *has Hausdorff dimension* $1/2$ *with probability 1.*

*Proof.* By the strong Markov property, conditional on $\tau_a = t$, the set $\{s \geq 0 : W(s) = a\}$ has the same distribution as $t + \mathcal{Z}_W$, and therefore the same almost sure Hausdorff dimension. By Proposition 5.24, $\tau_a < \infty$ a.s., and hence, with probability one, $\dim_H\{s \geq 0 : W(s) = a\} = \dim_H \mathcal{Z}_W = 1/2$. $\square$