Math 6810
(Probability and Fractals)

Spring 2016

Lecture notes

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Recommended reading: (Do not purchase these books before consulting with your instructor!)

1. *Real Analysis* by H. L. Royden (4th edition), Prentice Hall.
2. *Probability and Measure* by P. Billingsley (3rd edition), Wiley.
3. *Probability with Martingales* by D. Williams, Cambridge University Press.
4. *Fractal Geometry: Foundations and Applications* by K. Falconer (2nd edition), Wiley.

Chapter 4

Random fractals

This chapter is based on Falconer, Chapters 4 and 15, and a paper by B. Hunt.

4.1 The potential theoretic method

Proposition 4.1. *Let μ be a mass distribution on $\mathbb{R}^n$, $F \subset \mathbb{R}^n$ a Borel set, and $0 < c < \infty$ a constant. Suppose that*

$$\limsup_{r \downarrow 0} \frac{\mu(B(x,r))}{r^s} < c \qquad \text{for all } x \in F. \tag{4.1}$$

Then $\mathcal{H}^s(F) \geq \mu(F)/c$.

Proof. For $\delta > 0$, let

$$F_\delta := \{x \in F : \mu(B(x,r)) < cr^s \text{ for all } 0 < r \leq \delta\}.$$

Let $\{U_i\}$ be a δ-cover of F. If $x \in U_i \cap F_\delta$, then $U_i \subset B(x, |U_i|)$ and

$$\mu(U_i) \leq \mu(B(x, |U_i|)) < c|U_i|^s.$$

It follows that

$$\mu(F_\delta) \leq \sum_i \{\mu(U_i) : U_i \cap F_\delta \neq \emptyset\} \leq c \sum_i |U_i|^s.$$

Taking the infimum over all δ-covers gives $\mu(F_\delta) \leq c\mathcal{H}^s_\delta(F)$. But, since F_δ increases to F as $\delta \to 0$ by (4.1), we have $\mu(F) = \lim_{\delta\to 0} \mu(F_\delta)$, and so $\mu(F) \leq c\mathcal{H}^s(F)$. □

Definition 4.2. Let μ be a mass distribution on $\mathbb{R}^n$ and $s \geq 0$. The *s-potential* at a point $x \in \mathbb{R}^n$ due to μ is

$$\phi_s(x) := \int \frac{d\mu(y)}{|x-y|^s}, \tag{4.2}$$

and the *s-energy* of μ is

$$I_s(\mu) := \int \phi_s(x) d\mu(x) = \int\int \frac{d\mu(x)d\mu(y)}{|x-y|^s}, \tag{4.3}$$

where each integral is taken over $\mathbb{R}^n$.

Observe that if $I_s(\mu) < \infty$, then μ is nonatomic; that is, $\mu(\{x\}) = 0$ for each $x \in \mathbb{R}^n$. The connection between s-energy and Hausdorff measure and Hausdorff dimension is as follows:

Theorem 4.3. *Let $F \subset \mathbb{R}^n$. If there is a mass distribution μ on F with $I_s(\mu) < \infty$, then $\mathcal{H}^s(F) = \infty$ and therefore, $\dim_H F \geq s$.*

(There is a sort-of-converse to this theorem, which we will not need - see Falconer, Theorem 4.13(b).)

Proof. Let μ be a mass distribution on F, so $\mu(F) > 0 = \mu(\mathbb{R}^n \backslash F)$, and suppose $I_s(\mu) < \infty$. Define the set

$$F_1 := \left\{ x \in F : \limsup_{r \downarrow 0} \frac{\mu(B(x,r))}{r^s} > 0 \right\}.$$

Fix $x \in F_1$. Then we can find $\varepsilon > 0$ and a sequence $\{r_i\}$ decreasing to 0 such that $\mu(B(x,r_i)) \geq \varepsilon r_i^s$ for each i. Since $\mu(\{x\}) = 0$, we can find $0 < q_i < r_i$ small enough so that $\mu(A_i) \geq \frac{1}{2}\varepsilon r_i^s$, where A_i is the annulus $A_i = B(x,r_i) \backslash B(x,q_i)$. Taking a subsequence if necessary, we may assume $r_{i+1} < q_i$, so that the annuli A_i are disjoint. It follows that

$$\phi_s(x) = \int \frac{d\mu(y)}{|x-y|^s} \geq \sum_{i=1}^{\infty} \int_{A_i} \frac{d\mu(y)}{|x-y|^s} \geq \sum_{i=1}^{\infty} \frac{1}{2} \varepsilon r_i^s r_i^{-s} = \infty.$$

On the other hand, $I_s(\mu) = \int \phi_s(x) d\mu(x) < \infty$, so $\phi_s(x) < \infty$ for μ-almost every x. Though it's not clear if F_1 is measurable (a Borel set), there is certainly a Borel set E such that $F_1 \subset E$ and $\mu(E) = 0$, and by definition of F_1,

$$\limsup_{r \downarrow 0} \frac{\mu(B(x,r))}{r^s} = 0 \qquad \text{for all } x \in F \backslash E.$$

Hence, by Proposition 4.1,

$$\mathcal{H}^s(F) \geq \mathcal{H}^s(F \backslash E) \geq \mu(F \backslash E)/c = \mu(F)/c,$$

for every $c > 0$. Therefore, $\mathcal{H}^s(F) = \infty$. □

We can use Theorem 4.3 to obtain (almost-sure) lower bounds for the Hausdorff dimension of random fractals as follows. Let $(\Omega, \mathcal{F}, \mathrm{P})$ be a probability space, and suppose for each $\omega \in \Omega$ we have a fractal F_ω. If we can find for each ω a mass distribution μ_ω on F_ω such that

$$\int_\Omega I_s(\mu_\omega) d\mathrm{P}(\omega) = \int_\Omega \iint \frac{d\mu_\omega(x) d\mu_\omega(y)}{|x-y|^s} d\mathrm{P}(\omega) < \infty,$$

then it will follow that $I_s(\mu_\omega) < \infty$ for almost every ω, and so $\dim_H F_\omega \geq s$ for almost every ω, in other words, with probability one. In many practical applications a suitable change of variable and Fubini's theorem can be applied to compute or estimate the above triple integral.

4.2 Random Cantor sets

We will construct a "statistically self-similar" set analogous to the Cantor set by randomly choosing the contraction ratios at each stage of the construction. Let $0 < a \le b < \frac{1}{2}$ be constants. Let $(\Omega, \mathcal{F}, \mathrm{P})$ be a probability space on which is defined a collection of random variables $C_{i_1,\dots,i_k}$, $k \in \mathbb{N}$, $(i_1, \dots, i_k) \in \mathcal{I}_k := \{1, 2\}^k$, with the following properties:

(i) For each $k \in \mathbb{N}$ and $(i_1, \dots, i_k) \in \mathcal{I}_k$, the random variables $C_{i_1,\dots,i_k,j}$, $j = 1, 2$ are independent of $\mathcal{F}_k$, where

$$\mathcal{F}_k := \sigma(\{C_{i_1,\dots,i_l} : 1 \le l \le k,\ (i_1, \dots, i_l) \in \mathcal{I}_l\}).$$

(ii) For each $k \in \mathbb{N}$ the collection of random pairs $\{(C_{i_1,\dots,i_k,1}, C_{i_1,\dots,i_k,2}) : (i_1, \dots, i_k) \in \mathcal{I}_k\}$ is independent.

(iii) For each $k \in \mathbb{N}$ and $(i_1, \dots, i_k) \in \mathcal{I}_k$, $C_{i_1,\dots,i_k,1} \sim C_1$ and $C_{i_1,\dots,i_k,2} \sim C_2$.

(iv) $a \le C_j \le b$ a.s. for $j = 1, 2$.

Note that we do not require independence of $C_{i_1,\dots,i_k,1}$ and $C_{i_1,\dots,i_k,2}$.

Now define a collection of random intervals $\{I_{i_1,\dots,i_k}\}$ as follows: put $I_1 = [0, C_1]$ and $I_2 = [1 - C_2, 1]$. For $k \in \mathbb{N}$ and $(i_1, \dots, i_k) \in \mathcal{I}_k$, let $I_{i_1,\dots,i_k,1}$ and $I_{i_1,\dots,i_k,2}$ be subintervals of $I := I_{i_1,\dots,i_k}$ such that $I_{i_1,\dots,i_k,1}$ has the same left endpoint as I, $I_{i_1,\dots,i_k,2}$ has the same right endpoint as I, and $|I_{i_1,\dots,i_k,j}| = C_{i_1,\dots,i_k,j}|I|$ for $j = 1, 2$. We call $I_{i_1,\dots,i_k}$ a *basic interval at level* k. Let

$$E_k = \bigcup_{(i_1,\dots,i_k) \in \mathcal{I}_k} I_{i_1,\dots,i_k}, \qquad k \in \mathbb{N},$$

and

$$F := \bigcap_{k=1}^{\infty} E_k. \tag{4.4}$$

Note that for each k, the basic intervals of level k are disjoint, and as a result, F has the topological properties of a Cantor set (perfect, totally disconnected).

Theorem 4.4. *For the random Cantor set F described above, $\dim_H F = s$ with probability 1, where s is the unique positive solution of*

$$\mathrm{E}(C_1^s + C_2^s) = 1. \tag{4.5}$$

Proof. It is straightforward to check (using that C_1 and C_2 are bounded random variables) that $f(s) := \mathrm{E}(C_1^s + C_2^s)$ is continuous and strictly decreasing in s, with $f(0) = 2$ and $f(1) = \mathrm{E}(C_1 + C_2) \le 2b < 1$, so (4.5) has a unique solution.

Let $\mathcal{E}_k$ be the (finite) collection of intervals $I_{i_1,\dots,i_k}$, $(i_1, \dots, i_k) \in \mathcal{I}_k$, with $\mathcal{E}_0 := \{[0, 1]\}$. For $I = I_{i_1,\dots,i_k} \in \mathcal{E}_k$, write $I_L := I_{i_1,\dots,i_k,1}$ and $I_R := I_{i_1,\dots,i_k,2}$. For $s > 0$, we have

$$\begin{aligned} \mathrm{E}\left(|I_{i_1,\dots,i_k,1}|^s + |I_{i_1,\dots,i_k,2}|^s | \mathcal{F}_k\right) &= \mathrm{E}(C^s_{i_1,\dots,i_k,1} + C^s_{i_1,\dots,i_k,2})|I_{i_1,\dots,i_k}|^s \\ &= \mathrm{E}(C_1^s + C_2^s)|I_{i_1,\dots,i_k}|^s, \end{aligned}$$

where we first used (i) and then (iii). Summing over all the intervals in $\mathcal{E}_k$ gives

$$\mathrm{E}\left(\sum_{I\in\mathcal{E}_{k+1}} |I|^s \middle| \mathcal{F}_k\right) = \sum_{I\in\mathcal{E}_k} |I|^s\, \mathrm{E}(C_1^s + C_2^s). \tag{4.6}$$

Taking expectations on both sides we obtain

$$\mathrm{E}\left(\sum_{I\in\mathcal{E}_{k+1}} |I|^s\right) = \mathrm{E}\left(\sum_{I\in\mathcal{E}_k} |I|^s\right) \mathrm{E}(C_1^s + C_2^s). \tag{4.7}$$

If s is the solution of (4.5), then (4.6) reduces to

$$\mathrm{E}\left(\sum_{I\in\mathcal{E}_{k+1}} |I|^s \middle| \mathcal{F}_k\right) = \sum_{I\in\mathcal{E}_k} |I|^s,$$

which shows that the sequence (X_k) defined by $X_k = \sum_{I\in\mathcal{E}_k} |I|^s$ is a martingale.

Exercise: Show that the martingale (X_k) is L^2-bounded. (*Hint*: Show that $\mathrm{E}(X_{k+1}^2|\mathcal{F}_k) \le X_k^2 + a\gamma^k$, where $\gamma = \mathrm{E}(C_1^{2s} + C_2^{2s}) < 1$ and a is a constant.)

As a result, $X := \lim_{k\to\infty} X_k$ exists almost surely, and $\mathrm{E}(X) = \mathrm{E}(X_0) = 1$. We claim that $X > 0$ almost surely. Let $q = \mathrm{P}(X = 0)$. Since $X \ge 0$ and $\mathrm{E}(X) = 1$, $q < 1$. Now

$$X_k = \sum_{I\in\mathcal{E}_k, I\subset I_1} |I|^s + \sum_{I\in\mathcal{E}_k, I\subset I_2} |I|^s,$$

and the two random sums on the right are independent by (ii) (for all $k \ge 2$), and each tends to 0 with probability q, by the self-similarity of the construction. Thus $q = \mathrm{P}(X_k \to 0) = q^2$, and so $q = 0$, proving the claim. It follows that there are random variables M_1 and M_2 such that

$$0 < M_1 \le X_k = \sum_{I\in\mathcal{E}_k} |I|^s \le M_2 < \infty \quad \text{a.s. for all } k. \tag{4.8}$$

Given $\delta > 0$, $\mathcal{E}_k$ is a δ-cover of F for large enough k, and so $\mathcal{H}^s(F) \le M_2 < \infty$ with probability 1. Hence, $\dim_H F \le s$ almost surely.

For the lower bound we use the potential theoretic method. Let s again be the solution of (4.5). For $I \in \mathcal{E}_k$, define the random variable

$$\mu(I) := \lim_{j\to\infty} \sum \left\{|J|^s : J \in \mathcal{E}_j, J \subset I\right\}. \tag{4.9}$$

By the same argument as above, this limit exists, is $\mathcal{F}_k$-measurable, and $0 < \mu(I) < \infty$ almost surely. Furthermore, if $I \in \mathcal{E}_k$, then $\mu(I) = \mu(I_L) + \mu(I_R)$, so μ extends to a (random!) mass distribution on $[0, 1]$ with support in F. (The complete proof of this fact is rather involved and is omitted here.) In addition, we have

$$\mathrm{E}[\mu(I)|\mathcal{F}_k] = |I|^s, \qquad I \in \mathcal{E}_k. \tag{4.10}$$

Fix $0 < t < s$. We will estimate the t-energy of μ. For $x, y \in F$, there is a largest k such that x and y belong to the same basic interval $I \in \mathcal{E}_k$; denote this interval by $x \wedge y$. If $I \in \mathcal{E}_k$, then the subintervals I_L and I_R are separated by a gap of length at least $d|I|$, where $d = 1 - 2b > 0$. Thus, for any $I \in \mathcal{E}_k$,

$$\begin{aligned}\iint_{x \wedge y = I} |x-y|^{-t} d\mu(x) d\mu(y) &= 2 \int_{I_L} \int_{I_R} |x-y|^{-t} d\mu(x) d\mu(y) \\ &\leq 2d^{-t} |I|^{-t} \mu(I_L) \mu(I_R),\end{aligned}$$

and so

$$\begin{aligned}\mathrm{E}\left(\iint_{x \wedge y = I} |x-y|^{-t} d\mu(x) d\mu(y) \big| \mathcal{F}_{k+1} \right) &\leq 2d^{-t} |I|^{-t}\, \mathrm{E}[\mu(I_L) | \mathcal{F}_{k+1}]\, \mathrm{E}[\mu(I_R) | \mathcal{F}_{k+1}] \\ &\leq 2d^{-t} |I|^{-t} |I_L|^s |I_R|^s \\ &\leq 2d^{-t} |I|^{2s-t}.\end{aligned}$$

Here the first inequality uses that, conditionally on $\mathcal{F}_{k+1}$, $\mu(I_R)$ and $\mu(I_L)$ are independent because of assumption (ii); and the second inequality follows from (4.10). Taking expectations we obtain

$$\mathrm{E}\left(\iint_{x \wedge y = I} |x-y|^{-t} d\mu(x) d\mu(y) \right) \leq 2d^{-t}\, \mathrm{E}(|I|^{2s-t}).$$

Summing over $I \in \mathcal{E}_k$ and iterating (4.7), we get

$$\mathrm{E}\left(\sum_{I \in \mathcal{E}_k} \iint_{x \wedge y = I} |x-y|^{-t} d\mu(x) d\mu(y) \right) \leq 2d^{-t}\, \mathrm{E}\left(\sum_{I \in \mathcal{E}_k} |I|^{2s-t} \right) = 2d^{-t} \lambda^k,$$

where $\lambda := \mathrm{E}(C_1^{2s-t} + C_2^{2s-t}) < 1$, since $2s - t > s$. Finally, we can sum over k to obtain

$$\begin{aligned}\mathrm{E}\left(\int_F \int_F |x-y|^{-t} d\mu(x) d\mu(y) \right) &= \mathrm{E}\left(\sum_{k=0}^{\infty} \sum_{I \in \mathcal{E}_k} \iint_{x \wedge y = I} |x-y|^{-t} d\mu(x) d\mu(y) \right) \\ &\leq \sum_{k=0}^{\infty} 2d^{-t} \lambda^k < \infty.\end{aligned}$$

This implies that the t-energy of μ is finite almost surely, and hence, by Theorem 4.3, $\dim_H F \geq t$ a.s. Since $t < s$ was arbitrary, it follows that $\dim_H \geq s$ almost surely. □

Remark 4.5. Note that the proof does not tell us whether $\mathcal{H}^s(F) > 0$. The condition that there is a minimum gap between basic intervals is not necessary. (A version of the open set condition is enough.) But without this assumption, the proof is more involved.

Example 4.6. Let U be a uniformly distributed random variable on $(1/3, 2/3)$. Consider the construction of a random Cantor set F whereby for each basic interval $I = I_{i_1,\ldots,i_k}$, the middle portion of length $U_{i_1,\ldots,i_k}|I|$ is removed from I, where the collection $\{U_{i_1,\ldots,i_k} : k \in$

$\mathbb{N}_0, (i_1, \ldots, i_k) \in \mathcal{I}_k)\}$ is independent and each $U_{i_1,\ldots,i_k} \sim U$. This fits the framework of the above theorem, with $C_1 = C_2 = (1-U)/2$. Thus, $\dim_H F = s$, where s is the solution of

$$\mathrm{E}(C_1^s + C_2^s) = 2\,\mathrm{E}\left[\left(\frac{1-U}{2}\right)^s\right] = 1. \tag{4.11}$$

Exercise 4.7. Solve (4.11) (numerically). (Solution: $\dim_H F = s \doteq .4966$).

4.3 A random Weierstrass function

In this section we randomize the construction of the Weierstrass function from Chapter 3 by adding random phases as follows. Let $\theta_1, \theta_2, \ldots$ be independent uniform$(0, 2\pi)$ random variables. For constants $\lambda > 1$ and $1 < s < 2$, define the random function

$$W(x) = \sum_{n=0}^{\infty} \lambda^{(s-2)n} \sin(\lambda^n x + \theta_n), \qquad 0 \le x \le 1. \tag{4.12}$$

The following theorem is due to B. Hunt ("The Hausdorff dimension of graphs of Weierstrass functions", *Proc. Amer. Math. Soc.* 126 (1998), no. 3, 791–800). We present his proof with minor changes in notation.

Theorem 4.8. *With probability one,* $\dim_H \operatorname{Graph}(W) = s$.

The proof uses convolutions of densities. We need a definition and some lemmas.

Definition 4.9. The *convolution* of two functions f and g in $L^1(\mathbb{R})$ is the function

$$f * g(x) := \int_{\mathbb{R}} f(y)g(x-y)dy.$$

An easy exercise (using Fubini's theorem) shows that $f * g$ is well defined and in $L^1(\mathbb{R})$.

Lemma 4.10. *Let X and Y be independent random variables and suppose X is absolutely continuous with density f_X. Then $X+Y$ is absolutely continuous with density f_{X+Y}, and* $\sup_z f_{X+Y}(z) \le \sup_x f_X(x)$.

Proof. Let μ_Y denote the distribution of Y, F_X the c.d.f. of X, and F_{X+Y} the c.d.f. of $X+Y$. Then, by integrating over the half-plane $x + y \le z$ and using Fubini's theorem,

$$F_{X+Y}(z) = \int_{\mathbb{R}} F_X(z-y)d\mu_Y(y), \qquad z \in \mathbb{R}.$$

Since F_X is absolutely continuous, it now follows easily that F_{X+Y} is absolutely continuous also. (Check this!) Then by differentiating both sides of the above equation,

$$f_{X+Y}(z) = \int_{\mathbb{R}} f_X(z-y)d\mu_Y(y),$$

from which the second statement of the lemma follows immediately. □

Lemma 4.11. *If X and Y are independent absolutely continuous random variables with densities f_X and f_Y, then $X+Y$ has density $f_{X+Y} = f_X * f_Y$.*

Proof. Easy exercise. □

Lemma 4.12 (Young's inequality for convolutions)**.** *Let p, q and r be real numbers in $(1,\infty)$ such that $\frac{1}{p}+\frac{1}{q}=\frac{1}{r}+1$. If $f \in L^p$ and $g \in L^q$, then $f * g \in L^r$, and*

$$\|f * g\|_r \leq \|f\|_p \|g\|_q.$$

(For a proof, which uses a generalized Hölder inequality, see *proofwiki.org*.)

Exercise 4.13. Let $q \neq 0$, $a \in \mathbb{R}$, and let θ be a uniform$(0,2\pi)$ random variable. Show that the random variable $X = q\cos(a+\theta)$ has density

$$f_X(x) = \begin{cases} \frac{1}{\pi\sqrt{q^2-x^2}} & \text{if } |x| < |q|, \\ 0 & \text{if } |x| \geq |q|. \end{cases}$$

Finally, we recall the trigonometric identity

$$\sin x - \sin y = 2\cos\left(\frac{x+y}{2}\right)\sin\left(\frac{x-y}{2}\right). \tag{4.13}$$

Proof of Theorem 4.8. First, it follows just as in the proof of Theorem 3.42 that W is Hölder continuous with exponent $2-s$, and hence, by Proposition 3.41, $\dim_H \operatorname{Graph}(W) \leq s$.

For the lower bound we use the potential-theoretic method. Let μ be the random measure on $\mathbb{R}^2$, supported on Graph(W), defined by

$$\mu(A) := \mathcal{L}(\{x \in [0,1] : (x, W(x)) \in A\}), \qquad A \in \mathcal{B}(\mathbb{R}^2),$$

where $\mathcal{L}$ denotes Lebesgue measure on $[0,1]$. If $A_1, A_2, \dots$ are disjoint subsets of $\mathbb{R}^2$, then the sets $\{x : (x, W(x)) \in A_i\}$ are disjoint, so μ is indeed a measure, and $\mu(\operatorname{Graph}(W)) = 1$.

Fix $1 < t < s$. Our goal is to show that $\mathrm{E}(I_t(\mu)) < \infty$, which will imply, as in the proof of Theorem 4.4, that $\dim_H \operatorname{Graph}(W) \geq t$ a.s., and so, since t is arbitrary, $\dim_H \operatorname{Graph}(W) \geq s$ a.s. Here we let $\mathbf{x}$, $\mathbf{y}$ denote points in $\mathbb{R}^2$. By a change-of-variable and the Pythagorean theorem,

$$I_t(\mu) = \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \frac{d\mu(\mathbf{x})d\mu(\mathbf{y})}{|\mathbf{x}-\mathbf{y}|^t} = \int_0^1\int_0^1 \frac{dxdy}{\left((x-y)^2 + (W(x)-W(y))^2\right)^{t/2}},$$

so by Fubini's theorem,

$$\mathrm{E}(I_t(\mu)) = \int_0^1\int_0^1 \mathrm{E}\left[\left((x-y)^2 + (W(x)-W(y))^2\right)^{-t/2}\right] dx\, dy. \tag{4.14}$$

Let $\mathrm{E}_{x,y}$ denote the expectation in the above double integral. We will estimate $\mathrm{E}_{x,y}$ for all $x, y \in [0,1]$. Note first that, if $|x-y| \geq \pi/\lambda^2$, then $\mathrm{E}_{x,y} \leq (\lambda^2/\pi)^t$. Fix now x and y with

$0 < |x-y| < \pi/\lambda^2$, and let $Z = W(x) - W(y)$. Then Z is a random variable, and we wish to show that Z has a bounded density function $h(z)$ satisfying

$$h(z) \leq C|x-y|^{s-2}, \qquad z \in \mathbb{R}, \tag{4.15}$$

for some constant $C > 0$ that is independent of x and y. If we can show this, it will follow that

$$\begin{aligned}
\mathrm{E}_{x,y} = \mathrm{E}\left[\left((x-y)^2 + Z^2\right)^{-t/2}\right] &= \int_{-\infty}^{\infty} \frac{h(z)dz}{\left((x-y)^2+z^2\right)^{t/2}} \\
&\leq C\int_{-\infty}^{\infty} \frac{|x-y|^{s-2}dz}{\left((x-y)^2+z^2\right)^{t/2}} \\
&= C\int_{-\infty}^{\infty} \frac{|x-y|^{s-2}|x-y|dw}{|x-y|^t(1+w^2)^{t/2}} \\
&= C|x-y|^{s-1-t}\int_{-\infty}^{\infty} \frac{dw}{(1+w^2)^{t/2}},
\end{aligned}$$

where the next-to-last step uses the change-of-variable $z = |x-y|w$. Since $t > 1$, the integral in the last line above converges. And (check!) since $t < s$, the double integral

$$\int_0^1 \int_0^1 |x-y|^{s-1-t} dx dy$$

converges also. Thus, by (4.14), $\mathrm{E}(I_t(\mu)) < \infty$, as desired.

Using (4.13), we can write

$$\begin{aligned}
Z &= \sum_{n=0}^{\infty} \lambda^{(s-2)n}\big(\sin(\lambda^n x + \theta_n) - \sin(\lambda^n y + \theta_n)\big) \\
&= \sum_{n=0}^{\infty} 2\lambda^{(s-2)n} \sin\left(\lambda^n \frac{x-y}{2}\right) \cos\left(\lambda^n \frac{x+y}{2} + \theta_n\right) \\
&= \sum_{n=0}^{\infty} q_n \cos(r_n + \theta_n) =: \sum_{n=0}^{\infty} Z_n.
\end{aligned}$$

Note that the random variables $Z_1, Z_2, \ldots$ are independent, with Z_n having density

$$h_n(z) = \begin{cases} \frac{1}{\pi\sqrt{q^2-z^2}} & \text{if } |z| < |q|, \\ 0 & \text{if } |z| \geq |q|, \end{cases}$$

by Exercise 4.13. It follows from Lemma 4.10 that Z is absolutely continuous with density $h = h_0 * h_1 * h_2 \cdots$, and h is bounded by any upper bound for any finite convolution $h_j * \cdots * h_k$, where $j \leq k$.

Next, recall that $|x-y| < \pi/\lambda^2$. Thus, there is an integer $k \geq 2$ such that $\pi\lambda^{-k-1} < |x-y| \leq \pi\lambda^{-k}$. Fix this k. Then

$$\frac{\pi}{2\lambda^3} < \left|\lambda^{k-2}\frac{x-y}{2}\right| < \left|\lambda^k \frac{x-y}{2}\right| \leq \frac{\pi}{2},$$

and hence,

$$|q_n| > 2\sin\left(\frac{\pi}{2\lambda^3}\right)\lambda^{(s-2)k} > 2\sin\left(\frac{\pi}{2\lambda^3}\right)|x-y|^{2-s} \tag{4.16}$$

for $n = k-2, k-1, k$. Observe that $h_n \in L^p$ for $p < 2$, and by direct calculation, for $n = k-2, k-1, k$,

$$\|h_n\|_{3/2}^{3/2} = |q_n|^{-1/2} \int_{-1}^{1} \frac{dw}{\pi(1-w^2)^{3/4}} = K|q_n|^{-1/2},$$

so that $\|h_n\|_{3/2} = K^{2/3}|q_n|^{-1/3} \le K'|x-y|^{(s-2)/3}$ by (4.16), where K' depends only on λ. Now we apply first Young's inequality (Lemma 4.12) to obtain

$$\|h_{k-1} * h_k\|_3 \le \|h_{k-1}\|_{3/2}\|h_k\|_{3/2},$$

and then Hölder's inequality to conclude that

$$\begin{aligned} h_{k-2} * h_{k-1} * h_k(z) &= \int_{\mathbb{R}} h_{k-2}(z-x)(h_{k-1} * h_k)(x)dx \\ &\le \|h_{k-2}\|_{3/2}\|h_{k-1} * h_k\|_3 \\ &\le \|h_{k-2}\|_{3/2}\|h_{k-1}\|_{3/2}\|h_k\|_{3/2} \\ &\le K'^3|x-y|^{s-2}. \end{aligned}$$

But then this bound also applies to $h(z)$, and so we arrive at (4.15), completing the proof. $\square$