Math 6810 (Probability)

Fall 2012

Lecture notes

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Text: *Introduction to Stochastic Calculus with Applications*, by Fima C. Klebaner (3rd edition), Imperial College Press.

Other recommended reading: (Do not purchase these books before consulting with your instructor!)

1. *Real Analysis* by H. L. Royden (3rd edition), Prentice Hall.
2. *Probability and Measure* by Patrick Billingsley (3rd edition), Wiley.
3. *Probability with Martingales* by David Williams, Cambridge University Press.
4. *Stochastic Calculus for Finance I and II* by Steven E. Shreve, Springer.
5. *Brownian Motion and Stochastic Calculus* by Ioannis Karatzas and Steven E. Shreve, Springer. (Warning: this requires stamina, but is one of the few texts that is complete and mathematically rigorous)

Chapter 6

American options

Whereas a European option can be exercised only at the expiration date T, an American option is one that may be exercised at any time on or before the expiration date. We distinguish two types of options: finite-expiration options that have a finite deadline $T > 0$, and perpetual options, which have no deadline. For both types of American option, the exercise time will be a stopping time τ. (See Chapter 3!) The holder of the option can choose from among many stopping times, and thus the seller of the option must hedge his risk against all possible stopping times the buyer might employ. This is the fundamentally new idea in this chapter, and means that pricing an American option involves solving an *optimal stopping problem.*

6.1 Hitting time for a Brownian motion with drift

We will need the following result:

Theorem 6.1. *Let $W(t) : t \geq 0$ be a Brownian motion on a probability space $(\Omega, \mathcal{F}, \mathrm{P})$. Let $\mu \in \mathbb{R}$, and define $X(t) = W(t) + \mu t$. For $m > 0$, let*

$$\tau_m := \inf\{t \geq 0 : X(t) = m\},$$

where $\inf \emptyset = \infty$. Then, for all $\lambda > 0$,

$$\mathrm{E}(e^{-\lambda \tau_m}) = e^{-m(-\mu+\sqrt{\mu^2+2\lambda})}, \tag{6.1}$$

where we set $e^{-\infty} = 0$.

Exercise 6.2. Show that τ_m is a stopping time relative to the natural filtration $(\mathcal{F}_t)_t$ of $W(t)$. (Hint: by the continuity of $X(t)$, $\tau_m = \inf\{t \geq 0 : X(t) \geq m\}$. Recall that you have to show $\{\tau \leq t\} \in \mathcal{F}_t$ for every t. Again use the continuity of $X(t)$ and consider a countable dense set of time points, e.g. $\mathbb{Q} \cap [0, t]$.)

Proof of Theorem 6.1. Put $\sigma = -\mu + \sqrt{\mu^2 + 2\lambda}$; then $\sigma > 0$ and a direct calculation shows

$$\sigma\mu + \frac{1}{2}\sigma^2 = \lambda.$$

Thus,

$$e^{\sigma X(t) - \lambda t} = e^{\sigma W(t) - \sigma^2 t/2}, \tag{6.2}$$

and we know from Chapter 3 that this last process is a martingale. By the optimal stopping theorem (continuous time version!), the stopped martingale

$$M(t) := \exp\left\{\sigma W(t \wedge \tau_m) - \frac{1}{2}\sigma^2 (t \wedge \tau_m)\right\}$$

is also a martingale. Hence, for each $n \in \mathbb{N}$,

$$\begin{aligned} 1 = M(0) = \mathrm{E}\, M(n) &= \mathrm{E}\left[e^{\sigma X(n\wedge\tau_m) - \lambda(n\wedge\tau_m)}\right] \qquad \text{(by (6.2))} \\ &= \mathrm{E}\left[e^{\sigma m - \lambda\tau_m}\, \mathrm{I}_{\{\tau_m \le n\}}\right] + \mathrm{E}\left[e^{\sigma X(n) - \lambda n}\, \mathrm{I}_{\{\tau_m > n\}}\right]. \end{aligned}$$

Now the nonnegative r.v.'s $e^{\sigma m - \lambda\tau_m}\, \mathrm{I}_{\{\tau_m \le n\}}$ increase in n to the limit $e^{\sigma m - \lambda\tau_m}\, \mathrm{I}_{\{\tau_m < \infty\}}$, so by MCT,

$$\lim_{n\to\infty} \mathrm{E}\left[e^{\sigma m - \lambda\tau_m}\, \mathrm{I}_{\{\tau_m \le n\}}\right] = \mathrm{E}\left[e^{\sigma m - \lambda\tau_m}\, \mathrm{I}_{\{\tau_m < \infty\}}\right].$$

On the other hand,

$$0 \le e^{\sigma X(n) - \lambda n}\, \mathrm{I}_{\{\tau_m > n\}} \le e^{\sigma m - \lambda n},$$

and so

$$0 \le \mathrm{E}\left[e^{\sigma X(n) - \lambda n}\, \mathrm{I}_{\{\tau_m > n\}}\right] \le e^{\sigma m - \lambda n} \to 0.$$

Therefore, we obtain

$$1 = \mathrm{E}\left[e^{\sigma m - \lambda\tau_m}\, \mathrm{I}_{\{\tau_m < \infty\}}\right],$$

or equivalently,

$$\mathrm{E}\left[e^{-\lambda\tau_m}\, \mathrm{I}_{\{\tau_m < \infty\}}\right] = e^{-\sigma m} = e^{-m(-\mu + \sqrt{\mu^2 + 2\lambda})}, \tag{6.3}$$

for all $\lambda > 0$. □

Note: Letting $\lambda \downarrow 0$ in (6.3), we get by MCT,

$$\mathrm{P}(\tau_m < \infty) = \mathrm{E}\left(\mathrm{I}_{\{\tau_m < \infty\}}\right) = \lim_{\lambda\downarrow 0} e^{-m(-\mu + \sqrt{\mu^2 + 2\lambda})} = e^{m\mu - m|\mu|}, \tag{6.4}$$

so that

$$\mathrm{P}(\tau_m < \infty) = \begin{cases} 1, & \text{when } \mu \ge 0, \\ e^{-2m|\mu|}, & \text{when } \mu \le 0. \end{cases} \tag{6.5}$$

6.2 The perpetual American put option

We now consider the perpetual American put option, whose owner has the right to sell a share of stock at any time $t > 0$ for a fixed price K. Of course, no clairvoyance of the future is allowed, so the stock must be sold at a *stopping time* τ which takes values in $[0, \infty]$, where $\tau = \infty$ means the option is never exercised. We assume the Black-Scholes model, with a constant interest rate $r > 0$, constant mean rate of return α, and constant volatility $\sigma > 0$. The discounted value of the perpetual American put when exercised at time τ is

$$e^{-r\tau}(K - S(\tau)),$$

which is interpreted as 0 if $\tau = \infty$. We would like to proceed as before and consider the expectation of this under a risk-neutral measure, but there is a subtlety: The risk-neutral measure used in Chapter 5 was based on a fixed finite time horizon T through the relation $Q(A) = \mathrm{E}[Z(T)\,\mathrm{I}_A]$, where $Z(t)$ was an exponential martingale derived from the parameters of the stock price process. This construction is clearly inadequate here: we need a measure on all of $\mathcal{F}_\infty = \sigma(W(t) : 0 \le t < \infty)$, not just on $\mathcal{F}_T$ for some finite T. We deal with this issue first.

6.2.1 Construction of the risk-neutral measure

We shall construct Brownian motion as the coordinate mapping process $W(t, \omega) := \omega(t)$ on the canonical space $(\Omega, \mathcal{F}, \mathrm{P}) = (C[0, \infty), \mathcal{B}(C[0, \infty)), \text{ Wiener measure})$ (see Section 3.2.6). We let $\mathcal{F}_t = \sigma(W(s) : 0 \le s \le t)$ for $0 \le t < \infty$, and $\mathcal{F}_\infty = \mathcal{B}(C[0, \infty))$. We define a process $Z(t)$ as in the Girsanov theorem: let $\theta = (\alpha - r)/\sigma$ (a constant!), and set

$$Z(t) := \exp\left\{ -\int_0^t \theta dW(u) - \frac{1}{2}\int_0^t \theta^2 du \right\} = e^{-\theta W(t) - \theta^2 t/2}, \qquad 0 \le t < \infty.$$

We know from Chapter 3 that $Z(t)$ is a martingale with mean 1. (This does not require Novikov's theorem!) Now for every $T \in [0, \infty)$, define a probability measure Q_T on $\mathcal{F}_T$ by

$$Q_T(A) := \mathrm{E}[Z(T)\,\mathrm{I}_A], \qquad A \in \mathcal{F}_T.$$

Exercise 6.3. Show that, if $0 \le t \le T$ and $A \in \mathcal{F}_t$, then $\mathrm{Q}_T(A) = \mathrm{Q}_t(A)$. (Use the martingale property of $Z(t)$.)

The exercise implies that the family $(\mathrm{Q}_T)_{0 \le T < \infty}$ is consistent, so it defines a finitely additive set function Q on the algebra $\bigcup_{0 \le T < \infty} \mathcal{F}_T$ which satisfies $\mathrm{Q}(\emptyset) = 0$ and $\mathrm{Q}(\Omega) = 1$. It can be shown with some effort that this set function is in fact countably additive, so by Carathéodory's extension theorem it can be uniquely extended to a probability measure Q defined on $\mathcal{F}_\infty = \sigma(\bigcup_{0 \le T < \infty} \mathcal{F}_T)$.

Now the process

$$\widetilde{W}(t) := W(t) + \theta t, \qquad t \ge 0$$

is adapted to $(\mathcal{F}_t)_t$, and is a Brownian motion under Q. Note that the restriction of Q to $\mathcal{F}_T$ is Q_T, for $0 \leq T < \infty$. We write

$$\left.\frac{d\mathrm{Q}}{d\mathrm{P}}\right|_{\mathcal{F}_T} = Z(T).$$

However, as a measure on $\mathcal{F}_\infty$, Q is *not* absolutely continuous with respect to P, at least not if $\theta \neq 0$, as it usually is in real financial markets. For instance, the law of large numbers (Proposition 3.21) implies

$$\mathrm{Q}\left(\lim_{t\to\infty} \frac{W(t)}{t} = -\theta\right) = \mathrm{Q}\left(\lim_{t\to\infty} \frac{\widetilde{W}(t)}{t} = 0\right) = 1,$$

whereas

$$\mathrm{P}\left(\lim_{t\to\infty} \frac{W(t)}{t} = -\theta\right) = 0.$$

This shows that neither P nor Q is absolutely continuous with respect to the other as measures on $\mathcal{F}_\infty$.

We recall from Chapter 5 that in terms of $\widetilde{W}$, the SDE for the stock price becomes

$$dS(t) = rS(t)dt + \sigma S(t)d\widetilde{W}(t). \tag{6.6}$$

This has the explicit solution

$$S(t) = S(0)\exp\left\{\left(r - \frac{\sigma^2}{2}\right)t + \sigma\widetilde{W}(t)\right\}. \tag{6.7}$$

6.2.2 Price of the perpetual American put

We follow Shreve, Section 8.3. Assume the model setup of the previous subsection. Let $\mathcal{T}$ denote the collection of all stopping times τ (relative to the filtration $(\mathcal{F}_t)_t$) taking values in $[0, \infty]$.

Definition 6.4. The *price* of the perpetual American put option with strike K is defined by

$$v^*(x) = \sup_{\tau\in\mathcal{T}} \mathrm{E}_Q\left[e^{-r\tau}(K - S(\tau))\right], \tag{6.8}$$

when $S(0) = x$.

It will be explained later, by arbitrage considerations, why it makes sense to define the price of the option this way. But first we focus on the *optimal stopping problem* (6.8). We wish to find a specific stopping time τ^* that attains the supremum, that is,

$$\mathrm{E}_Q\left[e^{-r\tau^*}(K - S(\tau^*))\right] = v^*(x).$$

At each time t, the amount of time left is the same (i.e. infinity), so one would expect that at any time t, the decision whether to stop and exercise the option or to wait longer should depend only on the stock price $S(t)$, but not on t itself. A further look at (6.8) leads us to guess that the optimal stopping time might well be of the form

$$\tau_L := \inf\{t \geq 0 : S(t) \leq L\}, \tag{6.9}$$

for some constant $L < K$. We compute first the expectation in (6.8) for this stopping time τ_L. Put

$$v_L(x) := \mathrm{E}_Q \left[e^{-r\tau_L}(K - S(\tau_L)) \right],$$

when $S(0) = x$, and note that, by the continuity of $S(t)$,

$$v_L(x) = \begin{cases} K - x, & \text{if } 0 \leq x \leq L, \\ (K - L)\,\mathrm{E}_Q(e^{-r\tau_L}), & \text{if } x \geq L, \end{cases}$$

since $S(\tau_L) = L$ a.s. when $S(0) \geq L$. Using (6.7), we have that $S(t) = L$ if and only if

$$-\widetilde{W}(t) - \frac{1}{\sigma}\left(r - \frac{\sigma^2}{2}\right)t = \frac{1}{\sigma}\log\frac{x}{L}.$$

Thus we can apply Theorem 6.1, with $-\widetilde{W}$ instead of W, E_Q instead of E, and $\lambda = r$, $\mu = -\sigma^{-1}(r - \frac{1}{2}\sigma^2)$, and $m = \sigma^{-1}\log(x/L)$. This gives, after some algebra(!),

$$\mathrm{E}_Q(e^{-r\tau_L}) = \exp\left\{-\frac{1}{\sigma}\log\frac{x}{L}\cdot\frac{2r}{\sigma}\right\} = \left(\frac{x}{L}\right)^{-2r/\sigma^2},$$

when $S(0) = x \geq L$. Thus, we obtain the formula

$$v_L(x) = \begin{cases} K - x, & \text{if } 0 \leq x \leq L, \\ (K - L)\left(\frac{x}{L}\right)^{-2r/\sigma^2}, & \text{if } x > L. \end{cases} \tag{6.10}$$

It is a straightforward calculus exercise to maximize this over L. For each x, the maximum value of the second expression in the above equation is attained at

$$L^* = \frac{2r}{2r + \sigma^2}K.$$

(Note that indeed, $L^* < K$.) One can check that at $x = L^*$, the left- and right-hand derivatives of $v_{L^*}(x)$ are equal (both $= -1$). This could have been guessed from the figure below, which shows the graph of $v_L(x)$ for several values of L. The fact that, although the formula for $v_{L^*}(x)$ has two cases, the two pieces of the graph connect "smoothly" (i.e. with a common tangent line), is known as the "principle of smooth fit", which is a useful heuristic tool in the theory of optimal stopping.

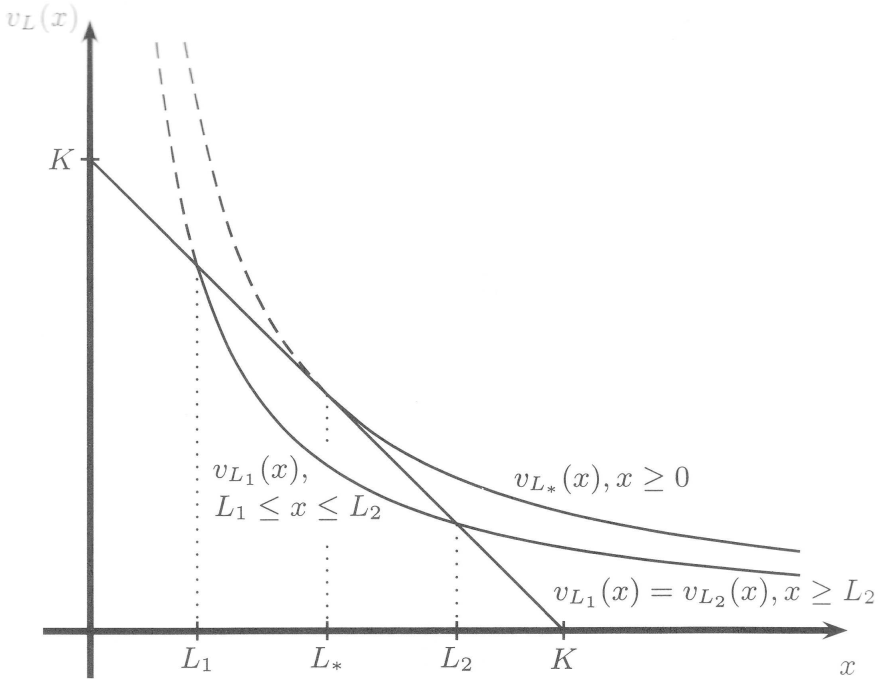

Fig. 8.3.1. $(K - L)\left(\frac{x}{L}\right)^{-\frac{2r}{\sigma^2}}$ for three values of L.

(Figure reproduced from Shreve, p. 350)

6.2.3 Optimality of the rule τ_{L^*}

Theorem 6.5. *The process* $e^{-rt}v_{L^*}(S(t))$ *is a supermartingale under* Q*, and the stopped process* $e^{-r(t\wedge\tau_{L^*})}v_{L^*}(S(t\wedge\tau_{L^*}))$ *is a martingale under* Q*.*

Proof. We first observe that v'_{L^*} is continuous, and v''_{L^*} is defined and continuous everywhere except at $x = L^*$. Furthermore, v_{L^*} satisfies the differential equation

$$rv_{L^*}(x) - rxv'_{L^*}(x) - \frac{1}{2}\sigma^2x^2v''_{L^*}(x) = \begin{cases} rK, & \text{if } 0 \le x < L^*, \\ 0, & \text{if } x > L^*. \end{cases} \tag{6.11}$$

(Check this!) We want to apply Itô's formula to the function $f(x,t) = e^{-rt}v_{L^*}(x)$. It can be shown that Itô's formula still holds when f_{xx} has jumps, as long as f_x is continuous. (This is proved using the concept of *local time*; see K & S, Section 3.6.) Thus, we obtain

$$\begin{aligned} d\left(e^{-rt}v_{L^*}(S(t))\right) &= df(S(t),t) \\ &= f_t(S(t),t)dt + f_x(S(t),t)dS(t) + \frac{1}{2}f_{xx}(S(t),t)d[S,S](t) \\ &= e^{-rt}\left[-rv_{L^*}(S(t)) + rS(t)v'_{L^*}(S(t)) + \frac{1}{2}\sigma^2S^2(t)v''_{L^*}(S(t))\right]dt \\ &\qquad + e^{-rt}\sigma S(t)v'_{L^*}(S(t))d\widetilde{W}(t), \end{aligned}$$

after substituting (6.6). Now it does not matter that $v''_{L^*}(L^*)$ is undefined, because $\mathrm{Q}(S(t) = L^*) = 0$ for all t, and so the set $\{t \ge 0 : S(t) = L^*\}$ has Lebesgue-measure zero with probability one. Using (6.11), the above differential simplifies to

$$d\left(e^{-rt}v_{L^*}(S(t))\right) = -e^{-rt}rK\,\mathrm{I}_{\{S(t)<L^*\}}\,dt + e^{-rt}\sigma S(t)v'_{L^*}(S(t))d\widetilde{W}(t). \tag{6.12}$$

In integral form:

$$e^{-rt}v_{L^*}(S(t)) = v_{L^*}(S(0)) - \int_0^t e^{-ru}rK\,\mathrm{I}_{\{S(u)<L^*\}}\,du + \int_0^t e^{-ru}\sigma S(u)v'_{L^*}(S(u))d\widetilde{W}(u). \tag{6.13}$$

Because $xv'_{L^*}(x)$ is bounded, the integrand in the Itô integral is bounded and so the Itô integral is a Q-martingale in t. Since the rest of the right hand side above is nonincreasing in t, this implies that $e^{-rt}v_{L^*}(S(t))$ is a supermartingale under Q. Finally, the stopped process satisfies (6.13) with $t\wedge\tau_{L^*}$ in place of t. But for $u \le \tau_{L^*}$ we have $S(u) \ge L^*$, so the Riemann integral vanishes and we have

$$e^{-r(t\wedge\tau_{L^*})}v_{L^*}(S(t\wedge\tau_{L^*})) = v_{L^*}(S(0)) + \int_0^{t\wedge\tau_{L^*}} e^{-ru}\sigma S(u)v'_{L^*}(S(u))d\widetilde{W}(u).$$

Thus, the stopped process is a martingale under Q. ☐

Corollary 6.6. *The stopping time* τ_{L^*} *is optimal in* (6.8)*. In other words,*

$$v_{L^*} = v^* = \max_{\tau \in \mathcal{T}} \mathrm{E}_Q \left[e^{-r\tau}(K - S(\tau)) \right],$$

where $x = S(0)$*. Thus,* v_{L^*} *is the price of the perpetual American put option.*

Proof. If a process $X(t)$ is a supermartingale, then for any stopping time τ the stopped process $X(t \wedge \tau)$ is also a supermartingale. (Compare Proposition 2.18 for the discrete-time case.) Thus, for any stopping time $\tau \in \mathcal{T}$, we have

$$v_{L^*}(x) = v_{L^*}(S(0)) \geq \mathrm{E}_Q \left[e^{-r(t \wedge \tau)} v_{L^*}\big(S(t \wedge \tau)\big) \right]. \tag{6.14}$$

Since v_{L^*} is bounded, BCT gives

$$v_{L^*}(x) \geq \mathrm{E}_Q \left[e^{-r\tau} v_{L^*}(S(\tau)) \right] \geq \mathrm{E}_Q \left[e^{-r\tau}\big(K - S(\tau)\big) \right],$$

where the last inequality follows since $v_{L^*}(x) \geq K - x$ for all x, as can be seen easily from (6.10). Since τ was arbitrary, we conclude

$$v_{L^*}(x) \geq \sup_{\tau \in \mathcal{T}} \mathrm{E}_Q \left[e^{-r\tau}(K - S(\tau)) \right].$$

On the other hand, the choice $\tau = \tau_{L^*}$ gives equality in (6.14), and BCT gives

$$v_{L^*}(x) = \mathrm{E}_Q \left[e^{-r\tau} v_{L^*}(S(\tau_{L^*})) \right].$$

But (provided $x \geq L^*$), $S(\tau_{L^*}) = L^*$ and $v_{L^*}(L^*) = K - L^*$, so

$$e^{-r\tau_{L^*}} v_{L^*}(S(\tau_{L^*})) = e^{-r\tau_{L^*}} \big(K - S(\tau_{L^*})\big).$$

This holds also if $x < L^*$, in which case $S(\tau_{L^*}) = S(0) = x$, and if $\tau_{L^*} = \infty$, in which case both sides equal 0. Hence,

$$v_{L^*}(x) = \mathrm{E}_Q \left[e^{-r\tau_{L^*}} \big(K - S(\tau_{L^*})\big) \right] \leq \sup_{\tau \in \mathcal{T}} \mathrm{E}_Q \left[e^{-r\tau}(K - S(\tau)) \right],$$

and as a result, $v_{L^*} = v^*$. □

6.2.4 Hedging the American put option

Why did we define the price of the perpetual American put the way we did? Consider an agent with initial capital $V(0) = v_{L^*}(S(0))$. Suppose this agent invests at each time t an amount $a(t) = v'_{L^*}(S(t))$ in the stock, and invests the remainder of his fortune in the money market account. In addition, the agent consumes cash at rate $C(t) = rK\,\mathrm{I}_{\{S(t) < L^*\}}$. Let $V(t)$ be the value of the agent's portfolio. Then

$$dV(t) = a(t)dS(t) + r\big(V(t) - a(t)S(t)\big)dt - C(t)dt,$$

and so

$$\begin{aligned} d\big(e^{-rt}V(t)\big) &= e^{-rt}\big(-rV(t)dt + dV(t)\big) \\ &= e^{-rt}\big(a(t)dS(t) - ra(t)S(t)dt - C(t)dt\big) \\ &= e^{-rt}\big(a(t)\sigma S(t)d\widetilde{W}(t) - C(t)dt\big). \end{aligned}$$

Substituting $a(t) = v'_{L^*}(S(t))$ and $C(t) = rK\,\mathrm{I}_{\{S(t)<L^*\}}$ and comparing with (6.12) we see that

$$d\big(e^{-rt}V(t)\big) = d\left(e^{-rt}v_{L^*}(S(t))\right).$$

Since $V(0) = v_{L^*}(S(0))$, this implies that $V(t) = v_{L^*}(S(t))$ for all t prior to exercise of the option. In particular, $V(t) \geq (K - S(t))^+$ for all t until the option is exercised, so the agent can pay off a short position in the option regardless of when it is exercised. If the option is exercised after the optimal time τ_{L^*}, the agent can even consume cash during periods when the stock price is below L^*. This shows that, if the option is sold for more than $v_{L^*}(S(0))$, the seller of the option has an arbitrage opportunity.

On the other hand, if the option is exercised at the optimal time τ_{L^*}, then $V(\tau_{L^*}) = v_{L^*}(S(\tau_{L^*})) = v_{L^*}(L^*) = K - L^* = K - S(\tau_{L^*})$, so the agent has exactly enough to pay off a short position in the option, and can not consume cash. Thus, if the option were sold for a lower price than $v_{L^*}(S(0))$, the seller would not be able to hedge the option. Hence, $v_{L^*}(S(0))$ is the arbitrage-free price.

6.3 The finite-expiration American put

While the perpetual American put option has a simple explicit pricing formula, it is not actually traded in real financial markets. By contrast, the American put option with a finite expiration date $T > 0$ *is* traded in real markets. One can develop a pricing theory for it analogous to that of the previous section, but this pricing theory will necessarily be more complicated since the price of the option will depend not only on the strike price K, but now also on T. In particular, one has to solve the optimal stopping problem

$$v^*(x,t) = \sup_{\tau\in\mathcal{T}, t\leq\tau\leq T} \mathrm{E}_Q\left[e^{-r(\tau-t)}(K - S(\tau))^+\big|S(t) = x\right],$$

where Q is the risk-neutral measure. Here the optimal strategy at time t (stop or continue) will depend both on the stock price and on the time $T - t$ remaining until the deadline. It is no longer possible to give an explicit expression for $v^*(x,t)$, but one can derive a PDE for it analogous to (6.11). As with the perpetual American put, the finite-expiration American put can be hedged by an investment-consumption portfolio which may consume cash if the buyer of the option exercises it after the optimal time. For the details, see Shreve, Section 8.4.

6.4 The finite-expiration American call

We now turn to the American call option, which gives its owner the right to buy a share of stock for a fixed price K any time on or before a deadline $T > 0$. For American options, there is no direct relationship between the price of a call option and the price of a put option (i.e. no put-call parity). Indeed, we shall see that the for a stock which pays no dividends (the model considered so far), the pricing of a finite-expiration American call is in a sense uninteresting, as the arbitrage-free price is the same as for the European call option. We shall consider the more general American derivative security, which pays its holder $h(S(t))$ upon exercise at time t. We call $h(S(t))$ the *intrinsic value* of the derivative security.

Let $0 < T < \infty$, let Q be a risk-neutral probability measure, and let the stock price $S(t)$ satisfy the SDE

$$dS(t) = rS(t)dt + \sigma S(t)d\widetilde{W}(t),$$

where $\widetilde{W}(t)$ is a Brownian motion under Q, and r is the constant interest rate.

Lemma 6.7. *Let $h : [0,\infty) \to \mathbb{R}$ be a nonnegative convex function satisfying $h(0) = 0$. Then $e^{-rt}h(S(t))$ is a submartingale under* Q.

Proof. Convexity implies that

$$h(\lambda x) \leq \lambda h(x), \qquad x \geq 0, \quad 0 \leq \lambda \leq 1. \tag{6.15}$$

We can now calculate, for $0 \leq u \leq t \leq T$:

$$\begin{aligned} \mathrm{E}_Q\left[e^{-r(t-u)}h(S(t))\middle|\mathcal{F}_u\right] &\geq \mathrm{E}_Q\left[h\left(e^{-r(t-u)}S(t)\right)\middle|\mathcal{F}_u\right] && \text{(by (6.15))} \\ &\geq h\left(\mathrm{E}_Q\left[e^{-r(t-u)}S(t)\middle|\mathcal{F}_u\right]\right) && \text{(conditional Jensen)} \\ &= h\left(e^{ru}\,\mathrm{E}_Q\left[e^{-rt}S(t)|\mathcal{F}_u\right]\right) \\ &= h\left(e^{ru}e^{-ru}S(u)\right) && \text{(since } e^{-rt}S(t) \text{ is a Q-martingale)} \\ &= h(S(u)). \end{aligned}$$

Multiplying by e^{-ru} we obtain

$$\mathrm{E}_Q[e^{-rt}h(S(t))|\mathcal{F}_u] \geq e^{-ru}h(S(u)).$$

Thus, $e^{-rt}h(S(t))$ is a submartingale under Q. □

Theorem 6.8. *Let $h : [0,\infty) \to \mathbb{R}$ be a nonnegative convex function satisfying $h(0) = 0$. The price of the American derivative security expiring at time T and having intrinsic value $h(S(t)), 0 \leq t \leq T$, is the same as the price of the European derivative security paying $h(S(T))$ at time T.*

Proof. By Lemma 6.7,

$$\mathrm{E}_Q\left[e^{-r(T-t)}h(S(T))\big|\mathcal{F}_t\right] \geq h(S(t)), \qquad 0 \leq t \leq T.$$

The left hand side is the value of the European derivative security at time t. Thus, at any time t it is at least as good to own the European derivative security as it is to exercise the American derivative security. In other words, the early exercise option is worthless, and the price of the American derivative security is the same as the price of the European derivative security. □

The finite-expiration American call option becomes nontrivial for models where the stock pays dividends, either at discrete times or as a continuous flow. See Shreve, Section 8.5.2 for a treatment of the discrete case.