Math 6810 (Probability)

Fall 2012

Lecture notes

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February 19, 2013

Text: *Introduction to Stochastic Calculus with Applications*, by Fima C. Klebaner (3rd edition), Imperial College Press.

Other recommended reading: (Do not purchase these books before consulting with your instructor!)

1. *Real Analysis* by H. L. Royden (3rd edition), Prentice Hall.

2. *Probability and Measure* by Patrick Billingsley (3rd edition), Wiley.

3. *Probability with Martingales* by David Williams, Cambridge University Press.

4. *Stochastic Calculus for Finance I and II* by Steven E. Shreve, Springer.

5. *Brownian Motion and Stochastic Calculus* by Ioannis Karatzas and Steven E. Shreve, Springer. (Warning: this requires stamina, but is one of the few texts that is complete and mathematically rigorous)

Chapter 5

Elements of asset pricing

5.1 Arbitrage: the basic idea

See Kleb, Section 11.1

5.2 Finite market models: no-arbitrage pricing

See Kleb, Section 11.2

5.3 Change of measure

The First Fundamental Theorem of Asset Pricing says that a market model does not admit arbitrage if (and only if) there is a probability measure Q equivalent to P, such that the discounted stock price $S(t)/\beta(t)$ is a martingale. Such a probability measure Q, if it exists, is called a *risk-neutral measure* or *equivalent martingale measure* (EMM). In the binomial model such a probability measure Q was easy to construct: All we had to do was choose the probability p of an up-step appropriately. For more sophisticated market models, however, it is less clear how to come up with a risk-neutral measure. The main purpose of this section is to introduce the Girsanov change-of-measure theorem, which ensures under certain conditions that a risk-neutral measure Q indeed exists.

5.3.1 The moment-generating function

Definition 5.1. The *moment-generating function* of a random variable X is the function

$$M_X(t) := \mathrm{E}\left(e^{tX}\right),$$

and is defined for those $t \in \mathbb{R}$ such that the expectation is finite.

Exercise 5.2. (a) Let X be standard normal. Show that

$$M_X(t) = e^{t^2/2}, \qquad t \in \mathbb{R}.$$

(b) Compute $M_Y(t)$ if $Y \sim \text{Normal}(\mu, \sigma^2)$. (Hint: write $Y = \sigma X + \mu$, and use (a).)

The moment-generating function derives its name from the fact that, if $M_X(t)$ is finite for all t in a neighborhood of 0, then

$$\text{E}(X^n) = \left. \frac{d^n}{dt^n} M_X(t) \right|_{t=0}.$$

(This follows once it is verified that one can differentiate inside the expectation.)

The usefulness of moment-generating functions is based on the following fundamental result:

Theorem 5.3. *Let X and Y be random variables, and suppose $M_X(t) = M_Y(t)$ (finite) for all t in an open neighborhood of 0. Then $X \sim Y$.*

Proof. See Billingsley, Section 22.   □

Remark 5.4. It is in general *not* true that, if X and Y have the same sequence of moments, then $X \sim Y$. This is because $M_X(t)$ may be infinite for all $t \neq 0$ even if all moments of X are finite. (**Exercise:** Construct an example to illustrate this!)

5.3.2   Change of measure for a normal random variable

We begin with a concrete example to illustrate the main idea. Let $\mu \in \mathbb{R}$. Let P_μ be the Normal$(\mu, 1)$ distribution on $\mathbb{R}$, and f_μ the corresponding density. Then we may write

$$d\,\text{P}_\mu(x) = f_\mu(x)dx.$$

Recall that

$$f_\mu(x) = \frac{1}{\sqrt{2\pi}} e^{-(x-\mu)^2/2} = f_0(x)e^{\mu x - \mu^2/2} = f_0(x)\Lambda(x),$$

where

$$\Lambda(x) = e^{\mu x - \mu^2/2}.$$

We can write the above relation between the densities as

$$d\,\text{P}_\mu(x) = \Lambda(x)d\,\text{P}_0(x),$$

or, formally dividing both sides by $d\,\text{P}_0(x)$,

$$\Lambda(x) = \frac{d\,\text{P}_\mu(x)}{d\,\text{P}_0(x)}. \tag{5.1}$$

Now check that, since $0 < \Lambda(x) < \infty$ for all x, we have

$$\text{P}_\mu(A) = 0 \quad \Longleftrightarrow \quad \text{P}_0(A) = 0, \qquad A \in \mathcal{B}(\mathbb{R}).$$

Thus, P_μ and P_0 are mutually absolutely continuous (in other words, *equivalent*). And (5.1) states that Λ is the Radon-Nikodym derivative of P_μ with respect to P_0.

More generally, we can say the following. Assume from now until the end of the section that we have fixed a probability measure space $(\Omega, \mathcal{F}, \text{P})$. Denote the Normal$(\mu, \sigma^2)$ distribution by $N(\mu, \sigma^2)$.

Theorem 5.5. *Let X be a $N(0,1)$ r.v. under* P. *Define the random variable $\Lambda = e^{\mu X - \mu^2/2}$, and define a measure* Q *by*

$$Q(A) := \int_A \Lambda \, d\,P = E_P(I_A \, \Lambda), \qquad A \in \mathcal{F},$$

or equivalently,

$$\frac{dQ}{dP}(\omega) = \Lambda(\omega), \qquad \omega \in \Omega.$$

Then Q *is a probability measure equivalent to* P, *and $X \sim N(\mu, 1)$ under* Q.

Proof. First, $Q(\Omega) = E_P(\Lambda) = 1$ by Exercise 5.2, so Q is a probability measure. Since $0 < \Lambda < \infty$, Q is equivalent to P. We now calculate:

$$E_Q\!\left(e^{tX}\right) = E_P\left(e^{tX}\Lambda\right) = E_P\left(e^{(t+\mu)X - \mu^2/2}\right)$$
$$= e^{-\mu^2/2} E_P\left(e^{(t+\mu)X}\right) = e^{-\mu^2/2} e^{(t+\mu)^2/2} = e^{t\mu + t^2/2}.$$

The last quantity is the moment-generating function of the Normal$(\mu, 1)$ distribution (see Exercise 5.2(b)). Hence, by Theorem 5.3, $X \sim N(\mu, 1)$ under Q. $\qquad \square$

Corollary 5.6 (Removal of normal mean)**.** *Let X be a $N(0,1)$ r.v. under* P, *and let $Y = X + \mu$. Then there is a probability measure* Q *equivalent to* P *such that $Y \sim N(0,1)$ under* Q. *Moreover,*

$$\frac{dQ}{dP} = e^{-\mu X - \mu^2/2}.$$

Proof. Apply Theorem 5.5 with $-\mu$ in place of μ. $\qquad \square$

Note that it's also possible to change the variance of a normal random variable by a suitable change of measure; see Kleb, Theorem 10.4.

5.3.3 Lévy's martingale characterization of Brownian motion

In the proof of the following theorem, the stochastic integral with respect to a general martingale occurs. For a simple process $X(t)$ and a square-integrable continuous martingale $M(t)$, the stochastic integral $\int_0^t X(s)dM(s)$ is defined the same way as the Itô integral $\int_0^t X(s)dW(s)$. If $M(t)$ is square integrable and $X(t)$ is a measurable adapted process, the integral $I(t) := \int_0^t X(s)dM(s)$ is defined as a limit in L^2 of integrals of simple processes approximating $X(t)$. As before, the process $I(t)$ is itself a square-integrable martingale when

$$E\left(\int_0^T X^2(s)d[M,M](s)\right) < \infty.$$

For continuous but not necessarily square-integrable martingales the definition is more involved, but the stochastic integral $I(t) = \int_0^t X(s)dM(s)$ can be defined for any process $X(t)$ satisfying the condition

$$\int_0^T X^2(s)d[M,M](s) < \infty \qquad \text{a.s.}$$

This will in particular be the case if $X(t)$ is continuous. The process $I(t)$ need not be a martingale, but it is always a *local martingale*; see K&S, Section 3.2.D.

Theorem 5.7 (Lévy's characterization of Brownian motion). *Let $M(t)$ be a continuous martingale with $M(0) = 0$ and with quadratic variation $[M, M](t) = t$ for all t. Then $M(t)$ is a Brownian motion.*

Sketch of proof. We check the properties in the definition of Brownian motion. Since $M(t)$ is continuous and $M(0) = 0$, it suffices to show that $M(t)$ has independent Gaussian increments with mean zero and the correct variance. This will be done using the moment-generating function and integration with respect to the martingale M.

Let $f(x, t)$ be a $C^{2,1}$ function. That is, f is twice continuously differentiable in x and once in t. It can be shown by extending the proof of Theorem 4.18 (see K&S, Theorems 3.3.3 and 3.3.6) that Itô's formula holds for functions of a general martingale; that is (using time s as the origin),

$$
\begin{aligned}
f(M(t), t) &= f(M(s), s) + \int_s^t f_x(M(u), u)dM(u) + \int_s^t f_t(M(u), u)du \\
&\quad + \frac{1}{2}\int_s^t f_{xx}(M(u), u)d[M, M](u) \\
&= f(M(s), s) + \int_s^t \left(f_t(M(u), u) + \frac{1}{2}f_{xx}(M(u), u) \right) du + \int_s^t f_x(M(u), u)dM(u)
\end{aligned}
$$

by the assumption $[M, M](t) = t$. Under appropriate integrability assumptions, the last term is a martingale in t and therefore has conditional expectation zero given $\mathcal{F}_s$. Assuming this for the moment, we obtain

$$
\mathrm{E}[f(M(t), t)|\mathcal{F}_s] = f(M(s), s) + \mathrm{E}\left[\int_s^t \left(f_t(M(u), u) + \frac{1}{2}f_{xx}(M(u), u) \right) du \Big| \mathcal{F}_s \right]. \quad (5.2)
$$

Now fix $a \in \mathbb{R}$ and define

$$
f(x, t) = \exp\left\{ ax - \frac{1}{2}a^2 t \right\}.
$$

Check that $f_t(x, t) + \frac{1}{2}f_{xx}(x, t) = 0$, so that the last term of (5.2) vanishes and

$$
\mathrm{E}\left[\exp\left\{ aM(t) - \frac{1}{2}a^2 t \right\} \Big| \mathcal{F}_s \right] = \exp\left\{ aM(s) - \frac{1}{2}a^2 s \right\}.
$$

This can be written as

$$
\mathrm{E}\left[\exp\left\{ a\big(M(t) - M(s)\big) \right\} \Big| \mathcal{F}_s \right] = e^{a^2(t-s)/2}.
$$

Since a was arbitrary, this shows that the increment $M(t) - M(s)$ is independent of $\mathcal{F}_s$ and is a normal random variable with mean zero and variance $t - s$. (Note: the right hand side is the mgf of the $N(0, t - s)$ distribution!) This establishes the remaining properties of Brownian motion, and as a result, M *is* a Brownian motion. $\qquad \square$

Remark 5.8. The integrability condition needed to ensure that $\int_0^t f_x(M(u), u)dM(u)$ is a martingale seems too strong. But in a rigorous proof the characteristic function is used instead of the moment-generating function, and the function f in the above proof is replaced by $f(x, t) = \exp(iax + a^2 t/2)$. (Itô's formula can be applied to the real and imaginary parts of f.) This function has bounded partial derivative f_x, so it satisfies the integrability condition and $\int_0^t f_x(M(u), u)dM(u)$ is a martingale.

We will later also need the following multi-dimensional version of Lévy's theorem. It is proved in a similar way.

Theorem 5.9 (Lévy's theorem in multiple dimensions). *Let $M_1, \dots, M_d$ be martingales relative to the same filtration $\mathcal{F}_t$. Assume that $M_i(0) = 0$ and $M_i(t)$ is continuous for each i, and that*

$$[M_i, M_j](t) = \begin{cases} t, & \text{if } i = j \\ 0, & \text{if } i \neq j. \end{cases}$$

Then $(M_1, \dots, M_d)$ is a d-dimensional Brownian motion. That is, M_i is a Brownian motion for each i, and $M_1, \dots, M_d$ are independent.

5.3.4 The Girsanov change-of-measure theorem

We are now ready to state and prove Girsanov's change-of-measure theorem, which shows how to "remove the drift" of a Brownian motion. First, we need a few lemmas.

Note that, if P and Q are probability measures and $Q \ll P$ with $dQ = \Lambda dP$, then (unconditional) expectations with respect to P and Q are related by

$$\mathrm{E}_Q(X) = \mathrm{E}_P(X\Lambda).$$

Conditional expectations are related as follows.

Lemma 5.10 (General Bayes formula). *Let P and Q be probability measures such that $Q \ll P$ with $dQ = \Lambda dP$. Let $\mathcal{G}$ be any sub-σ-algebra of $\mathcal{F}$. If X is Q-integrable, then $X\Lambda$ is P-integrable and*

$$\mathrm{E}_Q[X|\mathcal{G}] = \frac{\mathrm{E}_P[X\Lambda|\mathcal{G}]}{\mathrm{E}_P[\Lambda|\mathcal{G}]}.$$

Proof. See Kleb, Theorem 10.8. $\qquad\qquad\square$

Lemma 5.11. *Let $Z(t)$, $0 \leq t \leq T$ be a positive martingale under P with mean 1, and define a probability measure Q by*

$$\frac{dQ}{dP} = Z(T).$$

Then, for $s \leq t$ and any $\mathcal{F}_t$-measurable r.v. X,

$$\mathrm{E}_Q[X|\mathcal{F}_s] = \frac{1}{Z(s)} \mathrm{E}_P\left[XZ(t)|\mathcal{F}_s\right].$$

Proof. Note that Q is indeed a probability measure, since $Q(\Omega) = E_P(Z(T)) = 1$. Using the tower law for conditional expectation, compute

$$
\begin{aligned}
E_Q[X|\mathcal{F}_s] &= \frac{E_P[XZ(T)|\mathcal{F}_s]}{E_P[Z(T)|\mathcal{F}_s]} && \text{(by Lemma 5.10)} \\
&= \frac{1}{Z(s)} E_P[XZ(T)|\mathcal{F}_s] && \text{(since } Z(t) \text{ is a martingale)} \\
&= \frac{1}{Z(s)} E_P\big[\, E_P[XZ(T)|\mathcal{F}_t]|\mathcal{F}_s\big] && \text{(tower law)} \\
&= \frac{1}{Z(s)} E_P\big[X\, E_P[Z(T)|\mathcal{F}_t]|\mathcal{F}_s\big] && \text{(since } X \text{ is } \mathcal{F}_t\text{-measurable)} \\
&= \frac{1}{Z(s)} E_P[XZ(t)|\mathcal{F}_s] && \text{(again since } Z(t) \text{ is a martingale).}
\end{aligned}
$$

$\square$

Corollary 5.12. *Let $Z(t)$ and Q be as in Lemma 5.11. A process $M(t)$ is a Q-martingale if and only if $Z(t)M(t)$ is a P-martingale.*

Proof. Easy exercise. $\square$

Proposition 5.13. *Suppose $Q \ll P$ and $X_n \to_P X$. Then $X_n \to_Q X$.*

Proof. Let $A_n = \{|X_n - X| \geq \varepsilon\}$. Then $P(A_n) \to 0$ by hypothesis. But $Q(A_n) = E_P(\Lambda\, I_{A_n})$, where $\Lambda = \frac{dQ}{dP}$. Since Λ is P-integrable, the DCT implies $Q(A_n) \to 0$, which means $X_n \to_Q X$. $\square$

Corollary 5.14. *If P and Q are equivalent probability measures and X is a process whose quadratic variation exists under P, then X has the same quadratic variation under Q.*

Proof. This follows from the previous proposition, since quadratic variation is defined as a limit in probability. $\square$

We also need the following result, which gives a sufficient condition for the process $Z(t)$ in the Girsanov theorem below to be a martingale. Its proof is beyond the scope of this course, but may be found in K&S, Section 3.5.D.

Theorem 5.15 (Novikov)**.** *Let $X(t)$, $0 \leq t \leq T$ be an adapted process satisfying the SDE*

$$
dX(t) = \Theta(t)X(t)dW(t).
$$

If

$$
E\left(\exp\left\{\frac{1}{2}\int_0^T \Theta^2(t)dt\right\}\right) < \infty, \tag{5.3}
$$

then $X(t)$ is a martingale.

Theorem 5.16 (Girsanov change of measure). *Let $W(t), 0 \le t \le T$ be a Brownian motion under* P, *and $\Theta(t)$ an adapted process. Define*

$$Z(t) = \exp\left\{-\int_0^t \Theta(u)dW(u) - \frac{1}{2}\int_0^t \Theta^2(u)du\right\}, \tag{5.4}$$

and

$$\widetilde{W}(t) = W(t) + \int_0^t \Theta(u)du.$$

Assume that Θ satisfies the Novikov condition (5.3). Then the process $Z(t)$ is a P-martingale with $E_P(Z(T)) = 1$. Furthermore, the probability measure Q *defined by*

$$\frac{dQ}{dP} = Z(T)$$

is equivalent to P, *and $\widetilde{W}(t)$ is a Brownian motion under* Q.

Proof. An application of Itô's formula gives

$$dZ(t) = -\Theta(t)Z(t)dW(t), \tag{5.5}$$

so by Novikov's theorem, $Z(t)$ is a martingale. Integrating both sides of the above SDE we get

$$Z(t) = Z(0) - \int_0^t \Theta(u)Z(u)dW(u),$$

and since the Itô integral has mean zero, it follows that $EZ(t) = Z(0) = 1$ for $0 \le t \le T$. This proves the first part of the theorem.

To prove the statement about $\widetilde{W}$, we use the Lévy characterization of Brownian motion, Theorem 5.7. Note first that $\widetilde{W}(t)$ is clearly continuous in t. Its quadratic variation under P is

$$[\widetilde{W}, \widetilde{W}](t) = [W, W](t) = t,$$

since the process $\int_0^t \Theta(u)du$ is of bounded variation. By Corollary 5.14, the quadratic variation of $\widetilde{W}$ is t under Q as well. It remains to show that $\widetilde{W}$ is a Q-martingale. This will follow from Corollary 5.12 once we verify that $\widetilde{W}(t)Z(t)$ is a P-martingale. First, note that by (5.5),

$$d\widetilde{W}(t)dZ(t) = -\big(dW(t) + \Theta(t)dt\big)\Theta(t)Z(t)dW(t)$$
$$= -\Theta^2(t)Z(t)dtdW(t) - \Theta(t)Z(t)(dW(t))^2 = -\Theta(t)Z(t)dt.$$

So the Itô product rule together with (5.5) gives

$$d\big(\widetilde{W}(t)Z(t)\big) = \widetilde{W}(t)dZ(t) + Z(t)d\widetilde{W}(t) + d\widetilde{W}(t)dZ(t)$$
$$= -\widetilde{W}(t)\Theta(t)Z(t)dW(t) + Z(t)\big\{dW(t) + \Theta(t)dt\big\} - \Theta(t)Z(t)dt$$
$$= \big\{-\widetilde{W}(t)\Theta(t) + 1\big\}Z(t)dW(t).$$

Thus, the differential of $\widetilde{W}(t)Z(t)$ has no dt term. Hence, assuming an appropriate integrability condition is satisfied, $\widetilde{W}(t)Z(t)$ is a martingale under P, as required.

(*Note*: It is not clear that $\int_0^t \left(-\widetilde{W}(u)\Theta(u)+1\right)Z(u)dW(u)$ is a martingale, but it is certainly a local martingale. Lévy's theorem actually holds under the weaker assumption that $M(t)$ is a continuous local martingale, and Corollary 5.12 holds also for local martingales. Thus, we still get the desired conclusion that $\widetilde{W}$ is a Q-Brownian motion.)   □

5.4   Market models based on Brownian motion

Back to finance. We will now use the analytic tools developed above to describe conditions under which a market model based on Brownian motion does not admit arbitrage, and to correctly price financial derivatives such as call options.

5.4.1   The model

Assume our market consists of a risky asset (a stock), and a riskless asset (a money market account or a bond). The price of the stock is $S(t)$ and the price of the riskless asset is $\beta(t)$. We will assume that the stock price $S(t)$ satisfies the SDE

$$dS(t) = \alpha(t)S(t)dt + \sigma(t)S(t)dW(t), \qquad 0 \le t \le T, \tag{5.6}$$

where the mean rate of return $\alpha(t)$ and the volatility $\sigma(t)$ are adapted measurable processes. We assume that $P(\sigma(t) > 0 \,\forall\, t) = 1$. Observe that since the SDE is linear, we can explicitly write the solution $S(t)$ as

$$S(t) = S(0) \exp\left\{ \int_0^t \sigma(s)dW(s) + \int_0^t \left(\alpha(s) - \frac{1}{2}\sigma^2(s)\right) ds \right\}. \tag{5.7}$$

We also assume that we have an adapted interest rate process $R(t)$, and put

$$\beta(t) := \exp\left\{ \int_0^t R(s)ds \right\}.$$

Thus, $\beta(t)$ is the value (per \$1 invested) at time t of a savings account with variable interest rate $R(t)$, compounded continuously. For convenience, we shall write

$$D(t) := \frac{1}{\beta(t)} = \exp\left\{ -\int_0^t R(s)ds \right\}. \tag{5.8}$$

Exercise 5.17. Use Itô's formula to deduce that

$$dD(t) = -R(t)D(t)dt. \tag{5.9}$$

We think of $D(t)$ as a *discount process*, and $D(t)S(t)$ is the *discounted stock price*. Note that $D(t)$ is of bounded variation, and hence its value is more predictable than the stock price $S(t)$, which has positive quadratic variation $\sigma^2(t)S^2(t)$. (Intuitively, a change in interest rate will not immediately affect the value of a savings account, but will do so only over time; by contrast, a change in stock price is felt immediately by an investor holding the stock.)

Since $dD(t)dS(t) = 0$, the discounted stock price has differential

$$
\begin{aligned}
d\big(D(t)S(t)\big) &= D(t)dS(t) + S(t)dD(t) \\
&= \big(\alpha(t) - R(t)\big)D(t)S(t)dt + \sigma(t)D(t)S(t)dW(t) \\
&= \sigma(t)D(t)S(t)\big[\Theta(t)dt + dW(t)\big],
\end{aligned}
$$

where $\Theta(t)$ is the *market price of risk*,

$$
\Theta(t) = \frac{\alpha(t) - R(t)}{\sigma(t)}. \tag{5.10}
$$

Note that the second line of the above calculation shows that the mean rate of return of the discounted stock price is $\alpha(t) - R(t)$, and the volatility is again $\sigma(t)$, the same as for the undiscounted stock price.

Recall that we want the discounted stock price to be a martingale. Thus we use the Girsanov theorem (Theorem 5.16) with Θ given by (5.10). Note that this requires Θ to satisfy the Novikov condition (5.3). Thus we make the following assumption.

Technical Assumption 1: The market price of risk $\Theta(t) = \big(\alpha(t) - R(t)\big)/\sigma(t)$ satisfies the Novikov condition (5.3).

Girsanov's theorem gives an equivalent probability measure Q under which the process

$$
\widetilde{W}(t) = W(t) + \int_0^t \Theta(s)ds
$$

is a Brownian motion. Putting this into the calculation of $d(D(t)S(t))$ above we obtain

$$
d\big(D(t)S(t)\big) = \sigma(t)D(t)S(t)d\widetilde{W}(t). \tag{5.11}
$$

Thus, by Novikov's theorem, the discounted stock price $D(t)S(t)$ is a Q-martingale provided the following holds:

Technical Assumption 2: The volatility $\sigma(t)$ satisfies the Novikov condition

$$
E_Q\left(\exp\left\{\frac{1}{2}\int_0^T \sigma^2(t)dt\right\}\right) < \infty.
$$

The new measure Q is called the *risk-neutral measure* or *equivalent martingale measure* (EMM). Note that we can rewrite the SDE (5.6) in terms of $\widetilde{W}$ using (5.10):

$$dS(t) = R(t)S(t)dt + \sigma(t)S(t)d\widetilde{W}(t).$$

This expresses that under Q, the mean rate of return of the stock price is equal to the interest rate $R(t)$ (rather than $\alpha(t)$). Thus, under Q, the stock and money market account have the same mean rate of return; hence the name "risk-neutral measure".

5.4.2   Replicating strategies

A *portfolio* or *strategy* is a pair $(a,b) = \big((a(t), b(t)) : 0 \le t \le T\big)$ of adapted stochastic processes, where $a(t)$ denotes the number of shares of the stock and $b(t)$ the number of units of bond held at time t. We do not require $a(t)$ and/or $b(t)$ to be integer-valued. We do assume that they are left-continuous in t with right-hand limits (lcrr). (In the most general setting, $a(t)$ and $b(t)$ are assumed to be *predictable*, or *previsible*. Every lcrr process is predictable; see Kleb, Section 8.2.) The *value* of the portfolio (a, b) at time t is

$$V(t) = a(t)S(t) + b(t)\beta(t).$$

Definition 5.18. A portfolio (a, b) is *self-financing* if changes in the portfolio's value are due only to changes in the prices of the stock and bond; that is, if

$$dV(t) = a(t)dS(t) + b(t)d\beta(t), \tag{5.12}$$

or equivalently,

$$V(t) = V(0) + \int_0^t a(u)dS(u) + \int_0^t b(u)d\beta(u).$$

(Here the first integral is an Itô integral and the second one is a Riemann-Stieltjes integral.)

Informally, a portfolio is self-financing if no funds are withdrawn from the portfolio or injected into the portfolio from the outside at any time. The portfolio is continuously rebalanced: at each time t, some number of shares of stock is exchanged for shares of bond, or vice versa.

Definition 5.19. A strategy (a, b) is *admissible* if it is self-financing, $V(t) \ge 0$ for all t, and

$$\int_0^T \mathrm{E}_Q \big[a(t)\sigma(t)D(t)S(t)\big]^2 dt < \infty. \tag{5.13}$$

Exercise 5.20. Let (a, b) be a self-financing strategy. Use (5.12) and Itô's product rule to derive

$$d\big(D(t)V(t)\big) = a(t)d\big(D(t)S(t)\big), \tag{5.14}$$

and deduce from (5.11) that

$$d\big(D(t)V(t)\big) = a(t)\sigma(t)D(t)S(t)d\widetilde{W}(t). \tag{5.15}$$

Equation (5.14) expresses that in a self-financing strategy, the change in discounted portfolio value is due only to a change in the discounted stock price. This is because the change in bond price is neutralized by the discounting. More importantly, (5.15) shows that the discounted value process $D(t)V(t)$ is an Itô integral process under the risk-neutral measure Q. This process becomes a Q martingale if the integrability condition (5.13) is satisfied; in other words, if the strategy (a, b) is admissible. We have proved:

Theorem 5.21. *If the strategy (a, b) is admissible, then the discounted value process $D(t)V(t)$ is a martingale under the risk-neutral measure* Q.

Definition 5.22. A *contingent claim* (or simply *claim*) is an $\mathcal{F}_T$-measurable nonnegative random variable X such that $E_Q|X| < \infty$.

A contingent claim represents an agreement to pay X at time T. For a call option, $X = (S_T - K)^+$.

Definition 5.23. A *replicating strategy* for a claim X is an admissible strategy whose value process satisfies $V(T) = X$. (More precisely, $V(T, \omega) = X(\omega)$ for all $\omega \in \Omega$.)

5.4.3 Pricing under the risk-neutral measure

Suppose a replicating strategy (a, b) for a claim X exists, and let $V(t)$ be its value at time t. We claim that the "correct" (arbitrage-free) price of X at any time t prior to the expiration date T is $V(t)$. For:

1. If the price is more than $V(t)$, then the seller of the derivative security that pays X at time T can sell the security, use an amount $V(t)$ to buy the portfolio (a, b) and put the rest of the proceeds under a mattress. At time T, the seller will have X to deliver, fulfilling the terms of the contract. The money under the mattress is a riskless profit.

2. Vice versa, if the price is $C < V(t)$, then the buyer of the derivative can realize a riskless profit by pursuing the opposite strategy: Short-sell the portfolio (a, b) to generate revenue $V(t)$, use an amount C to buy the derivative and put the rest, $V(t) - C$ under a mattress. At expiration, the buyer will receive X from the seller of the derivative, which he can use to liquidate his short position in the portfolio (a, b), since $V(T) = X$. Again, the money under the mattress is a riskless profit.

We now derive a formula for the arbitrage-free price $V(t)$. Since $D(t)V(t)$ is a Q-martingale and $V(T) = X$, we have

$$D(t)V(t) = E_Q[D(T)V(T)|\mathcal{F}_t] = E_Q[D(T)X|\mathcal{F}_t].$$

Dividing by $D(t)$, which is $\mathcal{F}_t$-measurable, we obtain

$$V(t) = E_Q\left[\frac{D(T)}{D(t)}X\,\bigg|\,\mathcal{F}_t\right].$$

Remembering that $D(t) = \exp\{-\int_0^t R(s)ds\}$, this last equation can be written as

$$V(t) = \mathrm{E}_Q\left[e^{-\int_t^T R(s)ds}X\,\middle|\,\mathcal{F}_t\right], \qquad 0 \le t \le T. \tag{5.16}$$

In some simple cases, as in the Black-Scholes model examined in the next section, the conditional expectation above can be explicitly calculated. Usually, however, it must be estimated by Monte Carlo simulation.

Note that, if $(\mathcal{F}_t)_t$ is the natural filtration of the Brownian motion $W(t)$, then $\mathcal{F}_0 = \{\emptyset, \Omega\}$ and (5.16) gives

$$V(0) = \mathrm{E}_Q\left[e^{-\int_0^T R(s)ds}X\right].$$

This is the initial capital needed to hedge a short position in the derivative security whose terminal payoff is X.

Summary: If a replicating strategy for a claim X exists, then the arbitrage-free price of X at any time $0 \le t \le T$ is $V(t)$ and is given by the formula (5.16).

5.4.4   The Martingale Representation Theorem

The last subsection makes clear the importance of replicating strategies. But how do we know if a replicating strategy exists? The following theorem gives the answer.

Theorem 5.24 (Martingale Representation Theorem). *Let $(\mathcal{F}_t)_t$ be the natural filtration of the Brownian motion $W(t), 0 \le t \le T$, and let $M(t)$ be a martingale with respect to $(\mathcal{F}_t)_t$. Then there is an adapted lcrr process $\Gamma(t), 0 \le t \le T$ such that*

$$M(t) = M(0) + \int_0^t \Gamma(u)dW(u), \qquad 0 \le t \le T. \tag{5.17}$$

The proof of this theorem is beyond the scope of this course. Its statement, however, is of fundamental importance in the theory of derivative pricing. It guarantees the existence of replicating strategies. Before explaining how, observe first that the hypothesis that $(\mathcal{F}_t)_t$ is the natural filtration of the Brownian motion $W(t)$ says in effect that the only source of uncertainty in the model is the Brownian motion. Hence, there is only one source of uncertainty to be removed by hedging. Thus, intuitively, we ought to be able to hedge any contingent claim using a single stock.

In order to apply the Martingale Representation Theorem to our derivative pricing problem, we need to be able to replace the original Brownian motion W in (5.17) with the process $\widetilde{W}$, which is a Brownian motion under the risk-neutral measure Q. This is not completely trivial, because the filtration $(\mathcal{F}_t)_t$ is the natural filtration of W, not of $\widetilde{W}$.

Corollary 5.25. *Let $(\mathcal{F}_t)_t$ be the natural filtration of the Brownian motion W. Let $\widetilde{W}$ be the process given by the Girsanov theorem, and Q the risk-neutral measure. Let $\widetilde{M}(t)$ be a*

Q-*martingale with respect to* $(\mathcal{F}_t)_t$. *Then there is an adapted lcrr process* $\widetilde{\Gamma}(t), 0 \leq t \leq T$ *such that*

$$\widetilde{M}(t) = \widetilde{M}(0) + \int_0^t \widetilde{\Gamma}(u)d\widetilde{W}(u), \qquad 0 \leq t \leq T. \tag{5.18}$$

Proof. Let $Z(t)$ be the process defined by (5.4), and recall from (5.5) that $dZ(t) = -\Theta(t)Z(t)dW(t)$. Apply Itô's formula with $f(x) = 1/x$ to obtain (check!!)

$$d\left(\frac{1}{Z(t)}\right) = \frac{\Theta(t)}{Z(t)}dW(t) + \frac{\Theta^2(t)}{Z(t)}dt.$$

Put $M(t) = \widetilde{M}(t)Z(t)$, and recall from Corollary 5.12 that $M(t)$ is a P-martingale. Thus, by Theorem 5.24, there is an adapted lcrr process $\Gamma(t)$ such that (5.17) holds, in other words, $dM(t) = \Gamma(t)dW(t)$. In particular we have $d[M, 1/Z](t) = \Gamma(t)\Theta(t)/Z(t)$. We can now apply the Itô product rule to calculate

$$\begin{aligned} d\widetilde{M}(t) = d\left(M(t) \cdot \frac{1}{Z(t)}\right) &= M(t)d\left(\frac{1}{Z(t)}\right) + \frac{1}{Z(t)}dM(t) + d\left[M, \frac{1}{Z}\right](t) \\ &= \text{(some calculation -- do it!)} \\ &= \frac{\Gamma(t) + M(t)\Theta(t)}{Z(t)}d\widetilde{W}(t). \end{aligned}$$

Thus, we can put

$$\widetilde{\Gamma}(t) = \frac{\Gamma(t) + M(t)\Theta(t)}{Z(t)},$$

and have our desired representation. $\qquad\square$

Now assume we have a market model as before, where $(\mathcal{F}_t)_t$ is the filtration generated by the Brownian motion $W(t)$. In other words, the Brownian motion W is the only source of randomness in the model. Let X be a contingent claim, and define a process $E(t)$ by

$$E(t) := \mathrm{E}_Q[D(T)X|\mathcal{F}_t], \qquad 0 \leq t \leq T. \tag{5.19}$$

Then $E(t)$ is a martingale under Q (use the tower law!), so by Corollary 5.25 there is an adapted process $\widetilde{\Gamma}(t)$ such that

$$E(t) = E(0) + \int_0^t \widetilde{\Gamma}(u)d\widetilde{W}(u), \qquad 0 \leq t \leq T. \tag{5.20}$$

The previous subsection shows that in order for (a, b) to be a replicating portfolio for X, its value $V(t)$ must satisfy

$$D(t)V(t) = E(t) \tag{5.21}$$

for $0 \leq t \leq T$. From (5.15), we have

$$D(t)V(t) = V(0) + \int_0^t a(u)\sigma(u)D(u)S(u)d\widetilde{W}(u), \qquad 0 \leq t \leq T.$$

Thus, we can ensure (5.21) to hold by choosing initial capital $V(0) = E(0)$ and choosing $a(t)$ so that

$$a(t)\sigma(t)D(t)S(t) = \widetilde{\Gamma}(t), \qquad 0 \le t \le T,$$

in other words,

$$a(t) = \frac{\widetilde{\Gamma}(t)}{\sigma(t)D(t)S(t)}, \qquad 0 \le t \le T. \tag{5.22}$$

Note that this is always possible: the denominator in the above equation is strictly positive by the assumption on $\sigma(t)$ and the positivity of the discounted stock price $D(t)S(t)$.

Observe that we need not worry about the integrability condition on $a(t)$ in Definition 5.19. The only purpose of that condition was to ensure that $D(t)V(t)$ is a martingale. But if we choose $a(t)$ as above, then $D(t)V(t) = E(t)$, which *is* a martingale by definition.

Conclusion: If $(\mathcal{F}_t)_t$ is the natural filtration of the Brownian motion W, then all integrable contingent claims can be replicated (hedged). Thus, the market model is *complete*.

We now know that any derivative security can be successfully hedged and have a formula for calculating the arbitrage-free price of the derivative. But in order for the theory to be truly useful, we need an explicit formula, or at least a recipe, for calculating the hedge $a(t)$. Unfortunately, $a(t)$ is expressed in terms of $\widetilde{\Gamma}(t)$, and the Martingale Representation Theorem merely tells us that $\widetilde{\Gamma}(t)$ exists, but does not specify how to find it. Resolving this issue requires one more piece of theory, which is presented in the next subsection.

5.4.5 The Markov property

The result of this subsection is stated in more generality than needed for our finance application. Recall that a process $X(t)$ has the *Markov property* (or $X(t)$ is a *Markov process*) if

$$\mathrm{P}(X(t) \le y | \mathcal{F}_s) = \mathrm{P}(X(t) \le y | X(s)) \qquad \text{almost surely, for all } y \in \mathbb{R} \text{ and } t > s.$$

Theorem 5.26 (Markov property of solutions to SDEs)**.** *Let $X(t)$ be a solution of the SDE*

$$dX(t) = \mu(X(t), t)dt + \sigma(X(t), t)dW(t). \tag{5.23}$$

Then $X(t)$ has the Markov property.

The proof is technical and is not given here, but see K&S, Section 5.4.C. We just give a heuristic explanation. We can interpret (5.23) as saying that

$$X(t + \Delta t) - X(t) \approx \mu(X(t), t)\Delta t + \sigma(X(t), t)\left\{W(t + \Delta t) - W(t)\right\}.$$

If we had exact equality here, this would show that the distribution of $X(t+\Delta t)$ depends on $\mathcal{F}_t$ (i.e. the history up to time t) only through $X(t)$, since the increment $W(t + \Delta t) - W(t)$ is independent of $\mathcal{F}_t$. This is precisely the Markov property.

Every Markov process is characterized by its *transition probability function*

$$P(y, t, x, s) := \mathrm{P}(X(t) \leq y | X(s) = x).$$

It can be shown that if $X(t)$ is a solution of an SDE as above, then $P(y, t, x, s)$ is absolutely continuous in y; that is,

$$P(y, t, x, s) = \int_{-\infty}^{y} p(z, t, x, s) dz,$$

where $p(z, t, x, s)$ is the *transition density function*. (It is a density function in the variable z for fixed t, x and s.) The transition density of a solution $X(t)$ to an SDE satisfies the *forward and backward differential equations*; see Kleb, Section 5.8.

Exercise 5.27. Compute $p(y, t, x, s)$ in case $X(t)$ is Brownian motion: $X(t) = W(t)$.

In particular, if h is any Borel function, we have the formula

$$\mathrm{E}[h(X(T))|X(t) = x] = \int_{\mathbb{R}} h(y) p(y, T, x, t) dy. \tag{5.24}$$

We will write the left-hand-side more compactly as

$$\mathrm{E}^{t,x}[h(X(T))] := \mathrm{E}[h(X(T))|X(t) = x].$$

This expresses that when considering the future values of $h(X(T))$, we view the process as having started at time t with initial value $X(t) = x$. In most cases, the transition density $p(y, T, x, t)$ is sufficiently smooth and differentiating under the integral sign yields smoothness of the function $g(x, t) := \mathrm{E}^{t,x}[h(X(T))]$. This will allow us to apply Itô's formula to $g(x, t)$. And that, in turn, will yield an explicit hedge for claims of the form $X = h(S(T))$, where $S(t)$ is the stock price process, at least in the following important special case.

5.5 The Black-Scholes option-pricing formula

Recall our stock price model (5.6). In this section we consider the simplest case of the model, assuming that $\alpha(t) = \alpha$ and $\sigma(t) = \sigma > 0$ are constant. Thus,

$$dS(t) = \alpha S(t) dt + \sigma S(t) dW(t). \tag{5.25}$$

We also assume a constant interest rate, that is, $R(t) = r$ for all t. Then $D(t)$ becomes nonrandom, and

$$D(t) = e^{-rt}.$$

Observe that the process Θ used in the Girsanov theorem is now constant:

$$\Theta(t) = \frac{\alpha(t) - R(t)}{\sigma(t)} = \frac{\alpha - r}{\sigma} =: \Theta.$$

It therefore certainly satisfies the Novikov condition (5.3) (Technical Assumption 1), so the Girsanov change-of-measure theorem applies and yields a risk-neutral measure Q under which the process

$$\widetilde{W}(t) = W(t) + \Theta t, \qquad 0 \le t \le T$$

is a Brownian motion. Since $\sigma(t)$ is constant, Technical Assumption 2 is satisfied as well, so the discounted stock price $e^{-rt}S(t)$ is a Q-martingale. In terms of $\widetilde{W}$, the SDE for the stock price becomes

$$dS(t) = rS(t)dt + \sigma S(t)d\widetilde{W}(t).$$

This has the explicit solution

$$S(t) = S(0)\exp\left\{ \left(r - \frac{\sigma^2}{2} \right) t + \sigma\widetilde{W}(t) \right\}. \tag{5.26}$$

Now suppose we have a contingent claim X. The arbitrage-free price of X at time t is given by (5.16), which simplifies in this setting to

$$V(t) = e^{-r(T-t)}\, \mathrm{E}_Q[X|\mathcal{F}_t].$$

We now assume further that the payoff X depends only on the final price of the stock: $X = h(S(T))$ for some Borel function h. We call such a claim X a *terminal value claim*. By the Markov property of $S(t)$,

$$V(t) = e^{-r(T-t)}\, \mathrm{E}_Q[h(S(T))|\mathcal{F}_t] = e^{-r(T-t)}\, \mathrm{E}_Q[h(S(T))|S(t)],$$

so it is clear that there is a nonrandom function $v(x,t)$ such that $V(t) = v(S(t),t)$, since the only random variable that $V(t)$ depends on is $S(t)$. Thus we can write

$$v(x,t) = e^{-r(T-t)}\, \mathrm{E}_Q[h(S(T))|S(t) = x] = e^{-r(T-t)}\, \mathrm{E}_Q^{t,x}[h(S(T))].$$

We could in principle calculate this by finding the transition density of $S(t)$ and using (5.24). But simpler is to remember that under Q, $\widetilde{W}(T) - \widetilde{W}(t) \sim N(0, T-t)$ and is independent of $\widetilde{W}(t)$. Thus, from (5.26),

$$S(T) = S(t)\exp\left\{ \left(r - \frac{\sigma^2}{2} \right) (T-t) + \sigma\big(\widetilde{W}(T) - \widetilde{W}(t)\big) \right\}$$
$$= S(t)\exp\left\{ \left(r - \frac{\sigma^2}{2} \right) (T-t) + \sigma\sqrt{T-t}\, Z \right\},$$

where Z is a standard normal r.v. independent of $S(t)$. Putting $c = \sigma\sqrt{T-t}$ and $d = (r - \frac{\sigma^2}{2})(T-t)$, we can therefore write

$$\mathrm{E}_Q^{t,x}[h(S(T))] = \mathrm{E}_Q\left[h\big(xe^{cZ+d}\big) \right] = \int_{-\infty}^{\infty} h\big(xe^{cz+d}\big)\phi(z)dz,$$

in which $\phi(z)$ is the standard normal density,

$$\phi(z) = \frac{1}{\sqrt{2\pi}}e^{-z^2/2}.$$

We illustrate this with the call option, where

$$X = h(S(T)) = (S(T) - K)^+.$$

In this case, we get

$$\mathrm{E}_Q^{t,x}\left[(S(T) - K)^+\right] = \int_{-\infty}^{\infty}\left(xe^{cz+d} - K\right)^+\phi(z)dz = \int_{z_0}^{\infty}\left(xe^{cz+d} - K\right)\phi(z)dz,$$

where

$$z_0 = \frac{1}{c}\left[\log\left(\frac{K}{x}\right) - d\right].$$

Exercise 5.28. Finish the computation to obtain

$$v(x,t) = x\Phi(g(x,t)) - Ke^{-r(T-t)}\Phi\big(g(x,t) - \sigma\sqrt{T-t}\big), \tag{5.27}$$

where $\Phi(z) = \int_{-\infty}^{z}\phi(u)du$ is the standard normal c.d.f., and

$$g(x,t) = \frac{\log\frac{x}{K} + \left(r + \frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{T-t}}. \tag{5.28}$$

(Hint: separate the integral in two parts. Complete the square in the first part, and do a change of variable.)

5.5.1 Replicating portfolio

We next turn to the problem of hedging the call option; in other words, finding a replicating portfolio $(a(t), b(t))$. We will do this by applying Itô's formula to the function $v(x,t)$. The Black-Scholes formula (5.27) shows that $v(x,t)$ has the required smoothness (it is $C^{2,1}$). Recall that $d[S,S](t) = \sigma^2 S^2(t)dt$, so

$$dV(t) = dv(S(t),t) = v_x(S(t),t)dS(t) + v_t(S(t),t)dt + \frac{1}{2}v_{xx}(S(t),t)d[S,S](t)$$

$$= v_x(S(t),t)dS(t) + \left[v_t(S(t),t) + \frac{1}{2}\sigma^2 S^2(t)v_{xx}(S(t),t)\right]dt.$$

On the other hand, since the portfolio is self-financing,

$$dV(t) = a(t)dS(t) + b(t)d\beta(t).$$

Here $\beta(t) = e^{rt}$, so $d\beta(t) = r\beta(t)dt$. Substituting this gives

$$dV(t) = a(t)dS(t) + rb(t)\beta(t)dt.$$

Because $S(t)$ has positive quadratic variation we can match the $dS(t)$ terms, and obtain

$$a(t) = v_x(S(t), t).$$

(In words, the number of shares of stock in a replicating portfolio should always equal the partial derivative of the option price with respect to the stock price.) Similarly matching the dt terms gives

$$rb(t)\beta(t) = v_t(S(t), t) + \frac{1}{2}\sigma^2 S^2(t)v_{xx}(S(t), t),$$

from which we can solve for $b(t)$. Substituting the found expressions for $a(t)$ and $b(t)$ into the equation

$$V(t) = a(t)S(t) + b(t)\beta(t)$$

and multiplying both sides by r, we obtain the following second-order linear partial differential equation (PDE) for $v = v(x, t)$:

$$\frac{1}{2}\sigma^2 x^2 v_{xx} + rxv_x + v_t = rv. \tag{5.29}$$

This equation is called the *Black-Scholes-Merton* differential equation. It is valid for any terminal value claim $X = h(S(T))$. We have the terminal condition

$$v(x, T) = h(x) = (x - K)^+.$$

The terminal condition, of course, does depend on the claim.

Because the equation (5.29) is a second-order PDE in two variables, more boundary conditions must be specified in order for the solution to be unique; we will not go into this here.

Exercise 5.29. Show that for the call option, the replicating strategy has

$$a(t) = \Phi\big(g(S(t), t)\big), \qquad b(t) = -Ke^{-rT}\Phi\big(g(S(t), t) - \sigma\sqrt{T - t}\big),$$

where $g(x, t)$ is given by (5.28). (Note: both formulas in Kleb, eq. (11.46) are wrong!) Conclude (why?) that $a(t)$ satisfies the integrability condition (5.13), so the strategy (a, b) is indeed admissible. Observe that the replicating strategy is always long on stock (i.e. $a(t) > 0$) and short on bond (i.e. $b(t) < 0$). This means money is always being borrowed to buy the stock. Note also that $V(t) \geq 0$ is obvious from the formula (5.16).

Remark 5.30. The option price given by the Black-Scholes option-pricing formula (5.27) differs in practice from the actual market price. This is because our model with a constant volatility σ is too simple. Note that for fixed t, T and K, we can always choose σ so that $v(t, x)$ given by (5.27) matches the market price. This value of σ is called the *implied volatility*. Unfortunately, the implied volatility is different for options with different strikes K. In practice, the implied volatility is often a convex function of K (for fixed T and t), and its graph is referred to as the *volatility smile*.

A somewhat more realistic model with nonconstant volatility is the *constant elasticity of variance* (CEV) model

$$dS(t) = rS(t)dt + \sigma S^\delta(t)d\widetilde{W}(t),$$

where $0 < \delta < 1$. The volatility $\sigma(t) = \sigma S^{\delta-1}(t)$ is a decreasing function of $S(t)$, so the higher the price, the less volatile the stock becomes. One can choose δ so that the model gives a good fit for option prices across a range of strikes at a fixed expiration date. But in order to account for different expiration dates as well, we should let the volatility depend on both t and $S(t)$. That is, $\sigma(t) = \sigma(S(t), t)$ where $\sigma(x, t)$ is a nonrandom function.

5.5.2 Forwards and put-call parity

Recall that a *forward* is an agreement to sell a share of stock at a future time T for a fixed price K, agreed upon at time 0. It differs from a call option in that the buyer of the contract is obligated to buy the stock at the expiration date. Thus, its payoff at time T is $S(T) - K$. Let $f(x, t)$ denote the arbitrage-free price of the forward at time t when $S(t) = x$. Then

$$
\begin{aligned}
D(t)f(S(t), t) = D(t)V(t) &= \mathrm{E}_Q[D(T)\{S(T) - K\}|\mathcal{F}_t] \\
&= \mathrm{E}_Q[D(T)S(T)|\mathcal{F}_t] - K\,\mathrm{E}_Q[D(T)|\mathcal{F}_t] \\
&= D(t)S(t) - K\,\mathrm{E}_Q[D(T)|\mathcal{F}_t],
\end{aligned}
$$

because $D(t)S(t)$ is a Q-martingale. Thus

$$f(S(t), t) = S(t) - K\,\mathrm{E}_Q[D(T)/D(t)|\mathcal{F}_t].$$

In the Black-Scholes model we have $D(t) = e^{-rt}$, and the above formula simplifies to give

$$f(x, t) = x - Ke^{-r(T-t)}.$$

The *forward price* of the stock is the value of K that should be written in the contract in order that no money has to change hands at the time of inception. That is, $f(S(0), 0) = 0$. The above equation shows that the correct value is $K = e^{rT}S(0)$.

Next, recall that a *put option* is a contract that gives the right to sell a share of stock at time T for an agreed upon price K. The value at expiration is $(K - S(T))^+$. Let $c(x, t)$ be the price of the call option, and $p(x, t)$ the price of the put option at time t when $S(t) = x$. Since

$$x - K = (x - K)^+ - (x - K)^- = (x - K)^+ - (K - x)^+$$

and expectation is linear, we have the relationship

$$f(x, t) = c(x, t) - p(x, t).$$

Thus, no new calculation is needed to find the price of the put option. The above relationship is referred to as *put-call parity*.

5.6   Multidimensional market models

In this section, let $\mathbf{W}(t) = (W_1(t), \ldots, W_d(t))$ be a d-dimensional Brownian motion defined on a probabilty space $(\Omega, \mathcal{F}, \mathrm{P})$, and $(\mathcal{F}_t)_t$ the filtration generated by $\mathbf{W}$. Fix $T > 0$ and assume $\mathcal{F} = \mathcal{F}_T$, so $\mathbf{W}$ is the only source of randomness in our model. We follow Shreve, Section 5.4.

5.6.1   The model

Assume there are m stocks, whose prices satisfy the SDEs

$$dS_i(t) = \alpha_i(t)S_i(t)dt + S_i(t)\sum_{j=1}^{d}\sigma_{ij}(t)dW_j(t), \qquad i = 1, \ldots, m, \qquad (5.30)$$

where $\alpha_i(t)$ and $\sigma_{ij}(t)$ are adapted processes. As before, we assume $\sigma_{ij}(t) > 0$ a.s. (We make no assumption, for the moment, about the relation between m and d.)

Our first goal is to compute the quadratic covariations between the m stock prices. Set

$$\sigma_i(t) := \left(\sum_{j=1}^{d}\sigma_{ij}^2(t)\right)^{1/2}, \qquad i = 1, \ldots, m,$$

and define processes

$$B_i(t) := \sum_{j=1}^{d}\int_0^t \frac{\sigma_{ij}(u)}{\sigma_i(u)}dW_j(u), \qquad i = 1, \ldots, m.$$

Each $B_i(t)$, being a sum of stochastic integrals, is a continuous local martingale.

Exercise 5.31. Show that $[B_i, B_i](t) = t$ for $i = 1, \ldots, m$.

Thus, by Lévy's theorem (Theorem 5.7), each B_i is a Brownian motion. We can rewrite (5.30) in terms of B_i as

$$dS_i(t) = \alpha_i(t)S_i(t)dt + \sigma_i(t)S_i(t)dB_i(t), \qquad i = 1, \ldots, m,$$

and this shows that $\sigma_i(t)$ is the volatility of $S_i(t)$.

We first calculate the correlation between B_i and B_k for $i \neq k$. Recall that $[W_i, W_j](t) = 0$ when $i \neq j$, so from the definition of $B_i(t)$ we can easily calculate

$$d[B_i, B_k](t) = \rho_{ik}(t)dt,$$

where

$$\rho_{ik}(t) = \frac{1}{\sigma_i(t)\sigma_k(t)}\sum_{j=1}^{d}\sigma_{ij}(t)\sigma_{kj}(t).$$

Itô's product rule (in integral form) now gives

$$B_i(t)B_k(t) = \int_0^t B_i(u)dB_k(u) + \int_0^t B_k(u)dB_i(u) + \int_0^t \rho_{ik}(u)du.$$

Since B_i is a Brownian motion, we certainly have $\int_0^T \mathrm{E}\big(B_i^2(t)\big)dt < \infty$ for each i, so the Itô integrals appearing above have mean zero. Thus, taking expectations on both sides, we obtain

$$\mathrm{Cov}\big(B_i(t), B_k(t)\big) = \mathrm{E}\big(B_i(t)B_k(t)\big) = \int_0^t \mathrm{E}\big(\rho_{ik}(u)\big)du.$$

If the processes $\sigma_{ij}(t)$ and $\sigma_{kj}(t)$ are constant and nonrandom, then so is $\rho_{ik}(t)$, and in this case the last equation reveals that the correlation between $B_i(t)$ and $B_k(t)$ is simply ρ_{ik}. In the more general case, we call $\rho_{ik}(t)$ the *instantaneous correlation* between $B_i(t)$ and $B_k(t)$.

Finally, we calculate

$$\begin{aligned} d[S_i, S_k](t) &= \sigma_i(t)\sigma_k(t)S_i(t)S_k(t)d[B_i, B_k](t) \\ &= \rho_{ik}(t)\sigma_i(t)\sigma_k(t)S_i(t)S_k(t). \end{aligned}$$

5.6.2 Existence of a risk-neutral measure

Define a discount process $D(t)$ as in (5.8).

Definition 5.32. A probability measure Q is *risk-neutral* if Q $\sim$ P, and for each $i = 1, \ldots, m$, the discounted stock price $D(t)S_i(t)$ is a Q-martingale.

To establish the existence of a risk-neutral measure, we need a multidimensional version of the Girsanov theorem. For $\mathbf{x} = (x_1, \ldots, x_d) \in \mathbb{R}^d$, let $\|\mathbf{x}\| = \big(\sum_{i=1}^d x_i^2\big)^{1/2}$ denote the Euclidean norm of $\mathbf{x}$. Recall the notation

$$\int_0^t \mathbf{X}(u) \cdot d\mathbf{W}(u) = \sum_{j=1}^d \int_0^t X_j(u)dW_j(u)$$

for an adapted process $\mathbf{X}(t) = (X_1(t), \ldots, X_d(t))$.

Theorem 5.33 (Multidimensional Girsanov theorem)**.** *Let* $\mathbf{\Theta}(t) = (\Theta_1(t), \ldots, \Theta_d(t))$ *be a d-dimensional adapted process satisfying the Novikov condition*

$$\mathrm{E}_P\left(\exp\left\{\frac{1}{2}\int_0^T \|\mathbf{\Theta}(t)\|^2 dt\right\}\right) < \infty. \tag{5.31}$$

Define a process

$$Z(t) := \exp\left\{-\int_0^t \mathbf{\Theta}(u) \cdot d\mathbf{W}(u) - \frac{1}{2}\int_0^t \|\mathbf{\Theta}(u)\|^2 du\right\}, \qquad 0 \le t \le T. \tag{5.32}$$

Then $Z(t)$ is a P-martingale and $E(Z(T)) = 1$. Let Q be the measure defined by

$$\frac{d\mathrm{Q}}{d\mathrm{P}} = Z(T).$$

Then Q is a probability measure equivalent to P, and under Q, the process

$$\widetilde{\mathbf{W}}(t) := \mathbf{W}(t) + \int_0^t \mathbf{\Theta}(u)du, \qquad 0 \le t \le T$$

is a d-dimensional Brownian motion.

The proof is similar to that of Theorem 5.16. The main difference is the use of the multidimensional Lévy theorem (Theorem 5.9 instead of Theorem 5.7). We omit the details. Observe the surprising fact that, while the components of $\widetilde{\mathbf{W}}$ can be far from independent under P (since they all depend on $\mathbf{\Theta}$), they are independent under Q.

Back to the discounted stock prices $D(t)S_i(t)$. We calculate their differential as

$$d\big(D(t)S_i(t)\big) = D(t)\big[dS_i(t) - R(t)S_i(t)dt\big] \tag{5.33}$$

$$= D(t)S_i(t)\left[(\alpha_i(t) - R(t))dt + \sum_{j=1}^d \sigma_{ij}(t)dW_j(t)\right]. \tag{5.34}$$

We would like to be able to write this as

$$d\big(D(t)S_i(t)\big) = D(t)S_i(t)\sum_{j=1}^d \sigma_{ij}(t)\left[\Theta_j(t)dt + dW_j(t)\right], \tag{5.35}$$

for then we can apply the multidimensional Girsanov theorem to find a risk-neutral measure under which each $D(t)S_i(t)$ is a martingale. Comparing the two expressions for $d\big(D(t)S_i(t)\big)$ it is clear that the processes Θ_j must satisfy

$$\alpha_i(t) - R(t) = \sum_{j=1}^d \sigma_{ij}(t)\Theta_j(t), \qquad i = 1, \dots, m. \tag{5.36}$$

This is a system of m linear equations in d unkowns $\Theta_1(t), \dots, \Theta_d(t)$, which are called the *market price of risk equations*. Suppose a solution of (5.36) exists and satisfies (5.31). Then the multidimensional Girsanov theorem gives an equivalent probability measure Q and a d-dimensional Brownian motion $\widetilde{\mathbf{W}}$ under Q, such that

$$d\big(D(t)S_i(t)\big) = D(t)S_i(t)\sum_{j=1}^d \sigma_{ij}(t)d\widetilde{W}_j(t).$$

This shows that $D(t)S_i(t)$ is a local martingale under Q. It can be shown that, under the condition

$$E_Q \left(\exp \left\{ \frac{1}{2} \sum_{j=1}^{d} \int_0^T \sigma_{ij}^2(t)dt \right\} \right) < \infty, \qquad i = 1, \ldots, m, \qquad (5.37)$$

$D(t)S_i(t)$ is a proper Q-martingale. Thus, Q is a risk-neutral measure.

Definition 5.34. An *arbitrage* is a portfolio value process $V(t)$ satisfying $V(0) = 0$, and

$$P(V(T) \geq 0) = 1, \qquad P(V(T) > 0) > 0. \qquad (5.38)$$

Proposition 5.35. *If there is no solution to the market price of risk equations* (5.36), *then an arbitrage exists.*

We will not prove this proposition, but we illustrate it with an example.

Example 5.36. Let $m = 2$ and $d = 1$, and assume α_i and σ_{ij} are constant. So the model is:

$$dS_i(t) = \alpha_i S_i(t)dt + \sigma_i S_i(t)dW(t), \qquad i = 1, 2.$$

Assume also that the interest rate is constant: $R(t) = r > 0$. The market price of risk equations reduce to

$$\alpha_1 - r = \sigma_1 \theta,$$
$$\alpha_2 - r = \sigma_2 \theta.$$

These equations have a solution if and only if

$$\frac{\alpha_1 - r}{\sigma_1} = \frac{\alpha_2 - r}{\sigma_2}.$$

Suppose this equation does not hold, say

$$\frac{\alpha_1 - r}{\sigma_1} > \frac{\alpha_2 - r}{\sigma_2}.$$

Then we can arbitrage one stock against the other. Define

$$\mu := \frac{\alpha_1 - r}{\sigma_1} - \frac{\alpha_2 - r}{\sigma_2},$$

so $\mu > 0$. Consider a portfolio (a_1, a_2, b) consisting at any time t of $a_1(t)$ shares of stock 1, $a_2(t)$ shares of stock 2, and $b(t)$ shares of the money market account. The value of the portfolio is

$$V(t) = a_1(t)S_1(t) + a_2(t)S_2(t) + b(t)e^{rt}.$$

Specifically, we take

$$a_1(t) = \frac{1}{\sigma_1 S_1(t)}, \qquad a_2(t) = -\frac{1}{\sigma_2 S_2(t)}.$$

We choose

$$b(0) = \frac{1}{\sigma_2} - \frac{1}{\sigma_1}$$

so that the initial portfolio value is $V(0) = 0$ (check!), and for $0 < t \leq T$, we choose $b(t)$ in such a way that the portfolio is self-financing; that is,

$$dV(t) = a_1(t)dS_1(t) + a_2(t)dS_2(t) + b(t)d\big(e^{rt}\big).$$

Exercise 5.37. Show that the last four equations imply that $b(t)$ is the deterministic function

$$b(t) = \frac{1}{\sigma_2} - \frac{1}{\sigma_1} - \frac{e^{rt} - 1}{r}\left(\frac{\alpha_1}{\sigma_1} - \frac{\alpha_2}{\sigma_2}\right).$$

Show that the value process $V(t)$ is deterministic, and

$$V(t) = \frac{\mu}{r}(e^{rt} - 1).$$

Since $\mu > 0$, we have $V(t) > 0$ for all $t > 0$, so $V(t)$ is an arbitrage.

Let us return to our general model (5.30). Recall that $\beta(t) = \exp\{\int_0^t R(u)du\}$.

Definition 5.38. A *portfolio* or *strategy* is an $m+1$-dimensional adapted lcrr process $(\mathbf{a}, b)$ consisting of a vector $\mathbf{a}(t) = (a_1(t), \ldots, a_m(t))$ and a process $b(t)$, where $a_i(t)$ denotes the number of shares of stock i, $i = 1, \ldots, m$ and $b(t)$ denotes the number of shares of the money market account at time t. A portfolio $(\mathbf{a}, b)$ is *self-financing* if its value process

$$V(t) = \sum_{i=1}^{m} a_i(t)S_i(t) + b(t)\beta(t)$$

satisfies the SDE

$$dV(t) = \sum_{i=1}^{m} a_i(t)dS_i(t) + b(t)d\beta(t).$$

As in the one-dimensional case, we can calculate the differential of the discounted value of a self-financing portfolio to be (see Exercise 5.20):

$$d\big(D(t)V(t)\big) = \sum_{i=1}^{m} a_i(t)d\big(D(t)S_i(t)\big). \tag{5.39}$$

Now suppose a solution to the market price of risk equations (5.36) exists and satisfies (5.31). Assume also that (5.37) holds. Then there is a risk-neutral probability measure Q under which each discounted stock price $D(t)S_i(t)$ is a martingale. Under appropriate integrability conditions on $a_i(t)$, $i = 1, \ldots, m$, (5.39) shows that the discounted portfolio value $D(t)V(t)$ is then a Q-martingale also. We might interpret these statements as saying that under the risk-neutral measure Q, each stock has instantaneous mean rate of return $R(t)$, the same as the money market account, and hence so does the the entire portfolio.

Definition 5.39. A self-financing strategy $(\mathbf{a}, b)$ is *admissible* if the corresponding discounted portfolio value $D(t)V(t)$ is a Q-martingale.

A sufficient condition for $D(t)V(t)$ to be a Q-martingale is that

$$\int_0^T \mathrm{E}_Q \left[\sum_{i=1}^m a_i(t)D(t)S_i(t)\sigma_{ij}(t) \right]^2 dt < \infty, \qquad j = 1, \ldots, d. \tag{5.40}$$

Exercise 5.40. Prove this.

Theorem 5.41 (First Fundamental Theorem of Asset Pricing)**.** *If a market model has a risk-neutral probability measure, then it does not admit arbitrage.*

Proof. The proof is essentially the same as in the finite market model; see Kleb, Theorem 11.6. □

5.6.3 Replicating portfolio

Theorem 5.42 (Multidimensional Martingale Representation Theorem)**.** *Assume $(\mathcal{F}_t)_t$ is the filtration generated by the d-dimensional Brownian motion $\mathbf{W}(t)$. Let $M(t), 0 \leq t \leq T$ be a P-martingale with respect to $(\mathcal{F}_t)_t$. Then there is an adapted d-dimensional process $\mathbf{\Gamma}(t) = \big(\Gamma_1(t), \ldots, \Gamma_d(t)\big), 0 \leq t \leq T$ such that*

$$M(t) = M(0) + \int_0^t \mathbf{\Gamma}(u) \cdot d\mathbf{W}(u), \qquad 0 \leq t \leq T.$$

If in addition Q and $\widetilde{\mathbf{W}}$ are as in Theorem 5.33 and $\widetilde{M}(t), 0 \leq t \leq T$ is a Q-martingale, then there is an adapted d-dimensional process $\widetilde{\mathbf{\Gamma}}(t) = \big(\widetilde{\Gamma}_1(t), \ldots, \widetilde{\Gamma}_d(t)\big)$ such that

$$\widetilde{M}(t) = \widetilde{M}(0) + \int_0^t \widetilde{\mathbf{\Gamma}}(u) \cdot d\widetilde{\mathbf{W}}(u), \qquad 0 \leq t \leq T.$$

Suppose a risk-neutral probability measure Q exists. We wish to replicate (hedge) a claim X. Define a process $E(t)$ by (5.19). Then $E(t)$ is again a martingale under Q, so by Theorem 5.42, there are adapted processes $\widetilde{\Gamma}_1(t), \ldots, \widetilde{\Gamma}_d(t)$ such that

$$E(t) = E(0) + \sum_{j=1}^d \int_0^t \widetilde{\Gamma}_j(u)d\widetilde{W}_j(u), \qquad 0 \leq t \leq T. \tag{5.41}$$

The portfolio $(\mathbf{a}, b)$ will replicate the claim X if

$$D(t)V(t) = E(t), \qquad 0 \leq t \leq T.$$

Since

$$d\big(D(t)V(t)\big) = \sum_{j=1}^d \sum_{i=1}^m a_i(t)D(t)S_i(t)\sigma_{ij}(t)d\widetilde{W}_j(t),$$

it follows that we should take $V(0) = E(0)$ and choose $a_i(t)$, $i = 1, \ldots, m$ so as to satisfy the *hedging equations*

$$\sum_{i=1}^{m} a_i(t) S_i(t) \sigma_{ij}(t) = \frac{\widetilde{\Gamma}_j(t)}{D(t)}, \qquad j = 1, \ldots, d. \tag{5.42}$$

This is a system of d linear equations in the m unknowns $a_1(t), \ldots, a_m(t)$.

5.6.4   Completeness of the market model

Recall that a market model is complete if every derivative security can be hedged.

Theorem 5.43 (Second Fundamental Theorem of Asset Pricing). *Consider a market model with a risk-neutral probability measure. The model is complete if and only if the risk-neutral probability measure is unique.*

Proof. Assume first that the model is complete. Suppose, by way of contradiction, that there are two risk-neutral measures Q_1 and Q_2. Let $A \in \mathcal{F}$ and recall our assumption that $\mathcal{F} = \mathcal{F}_T$. Consider the claim

$$X = \frac{I_A}{D(T)}.$$

Since the model is complete, there is a replicating portfolio for this claim X with value process $V(t)$, and $D(t)V(t)$ is a martingale under both Q_1 and Q_2. But $D(T)V(T) = D(T)X = I_A$, so

$$V(0) = D(0)V(0) = E_{Q_i}[D(T)V(T)] = E_{Q_i} I_A = Q_i(A), \qquad i = 1, 2.$$

Hence $Q_1(A) = Q_2(A)$. Since $A \in \mathcal{F}$ was arbitrary, this means Q_1 and Q_2 are the same measures.

Conversely, suppose there is only one risk-neutral measure. It can be shown that the only risk-neutral measures are those obtained via the Girsanov theorem by solving the market price of risk equations. (We omit the details here.) Thus, the system (5.36) must have a unique solution. We can write this system in the form $A\mathbf{x} = \mathbf{b}$, where

$$A = [\sigma_{ij}(t)]_{i=1, j=1}^{m, d}, \qquad \mathbf{x} = \begin{bmatrix} \Theta_1(t) \\ \vdots \\ \Theta_d(t) \end{bmatrix}, \qquad \mathbf{b} = \begin{bmatrix} \alpha_1(t) - R(t) \\ \vdots \\ \alpha_m(t) - R(t) \end{bmatrix}.$$

Since the system has a unique solution, we must have $m \geq d$ and A must have full rank (a pivot in every column). To show that the model is complete we must show that there is a solution to the hedging equations (5.42). These can be written as $A^T \mathbf{y} = \mathbf{c}$, where

$$\mathbf{y} = \begin{bmatrix} a_1(t) S_1(t) \\ \vdots \\ a_m(t) S_m(t) \end{bmatrix}, \qquad \mathbf{c} = \begin{bmatrix} \widetilde{\Gamma}_1(t)/D(t) \\ \vdots \\ \widetilde{\Gamma}_d(t)/D(t) \end{bmatrix}.$$

Since A has full rank, so does A^T and A^T has no more rows than columns. Thus, A^T has a pivot in every row and the equation $A^T \mathbf{y} = \mathbf{c}$ has a solution no matter what $\mathbf{c}$ is. Hence, every claim can be hedged, and the market model is complete. $\qquad \square$