Math 5810, Review for Exam 2, Solutions

1. a) $\left(\frac{9}{10}\right)^5$ b) $\frac{1}{1/10} = 10$ c) $20 \cdot \frac{1}{10} = 2$ d) $\binom{19}{2}\left(\frac{1}{10}\right)^3\left(\frac{9}{10}\right)^{17}$

2. a) $P(X \le \frac{1}{2}) = \int_0^{1/2} 12x^2(1-x)\,dx = \int_0^{1/2}(12x^2 - 12x^3)\,dx = \left[4x^3 - 3x^4\right]_0^{1/2}$
$= 4\cdot\left(\frac{1}{2}\right)^3 - 3\left(\frac{1}{2}\right)^4 = \frac{1}{2} - \frac{3}{16} = \frac{5}{16}.$

   b) $P(X = \frac{1}{3}) = 0$

   c) $E(X) = \int_0^1 12x^3(1-x)\,dx = \int_0^1 (12x^3 - 12x^4)\,dx = \left[3x^4 - \frac{12}{5}x^5\right]_0^1$
$= 3 - \frac{12}{5} = \frac{3}{5}$

   d) $E(X^2) = \int_0^1 12x^4(1-x)\,dx = \int_0^1 (12x^4 - 12x^5)\,dx = \left[\frac{12}{5}x^5 - 2x^6\right]_0^1$
$= \frac{12}{5} - 2 = \frac{2}{5} \Rightarrow Var(X) = \frac{2}{5} - \left(\frac{3}{5}\right)^2 = \frac{1}{25}.$

3. $P(X = k) = P(k \le T < k+1) = P(T \ge k) - P(T \ge k+1) = e^{-\lambda k} - e^{-\lambda(k+1)}$
$= e^{-\lambda k}(1 - e^{-\lambda}) = (e^{-\lambda})^k(1 - e^{-\lambda})$, so X has a (shifted) geometric distribution on $\{0,1,2,\dots\}$ with parameter $p = 1 - e^{-\lambda}$.

4. First, since Range $(T) = (0,\infty)$, Range $(Y) = (0,1)$. For $0 < y < 1$,
$F_Y(y) = P(Y \le y) = P(e^{-T} \le y) = P(-T \le \log y) = P(T > -\log y)$
$= e^{-\lambda(-\log y)} = e^{\lambda \log y} = (e^{\log y})^\lambda = y^\lambda$, and $f_Y(y) = F_Y'(y) = \lambda y^{\lambda-1}$!

5. Assume the weights of the hotel guests are independent of each other and all have the same distribution. Let X_i = weight of the i^{th} guest in the elevator, $i = 1,\dots,30$, and let $S_{30} := X_1 + \dots + X_{30}$ be the total weight. Then $E(S_{30}) = (30)(150) = 4500$ and $SD(S_{30}) = \sqrt{30}\,(55) = 301.25$. By the Central Limit Theorem,
$$P(S_{30} > 5000) \approx 1 - \Phi\left(\frac{5000-4500}{301.25}\right) = 1 - \Phi(1.66) = 1 - .9515 = \boxed{.0485}$$
Note: Continuity correction makes little sense here (why?), but also does little harm.

6. Let T be the number of tosses until the first head shows. Then $T \sim$ geometric (p). Let A = "A tosses the first head", B = "B tosses the first head". Then $P(A) = P(T \text{ is odd}) = \sum_{n=0}^{\infty} P(T = 2n+1) = \sum_{n=0}^{\infty} q^{2n}p = \frac{p}{1-q^2} = \frac{1}{1+q}$
and $P(B) = 1 - P(A) = \frac{q}{1+q} \le P(A)$, with equality if and only if $q = 1$ (so $p = 0$).

7. Let A_i = "the i^{th} letter is placed in the correct envelope", $i = 1,\dots,n$. Let N be the number of letters placed in the correct envelopes. Then $P(A_i) = \frac{1}{n}$ for each i, and $N = I_{A_1} + \dots + I_{A_n}$. Thus,
$$E(N) = P(A_1) + \dots + P(A_n) = n \cdot \frac{1}{n} = 1.$$

8. Let N be the total number of misprints, and X the number of misprints found. Then $N \sim \text{Poisson}(\mu)$, and $(X|N=n) \sim \text{Binomial}(n,p)$. So for $k = 0,1,2,\ldots,$

$$
\begin{aligned}
P(X=k) &= \sum_{n=k}^{\infty} P(X=k, N=n) = \sum_{n=k}^{\infty} P(X=k|N=n)\,P(N=n) \\
&= \sum_{n=k}^{\infty} \binom{n}{k} p^k (1-p)^{n-k} e^{-\mu} \frac{\mu^n}{n!} \\
&= e^{-\mu} \cdot p^k \cdot \frac{1}{k!} \sum_{n=k}^{\infty} \frac{(1-p)^{n-k}}{(n-k)!} \cdot \mu^n \\
&= e^{-\mu} \frac{(\mu p)^k}{k!} \sum_{n=k}^{\infty} \frac{[\mu(1-p)]^{n-k}}{(n-k)!} \\
&= e^{-\mu} \frac{(\mu p)^k}{k!} \cdot \sum_{j=0}^{\infty} \frac{[\mu(1-p)]^j}{j!} \\
&= e^{-\mu} \frac{(\mu p)^k}{k!} \cdot e^{\mu(1-p)} \\
&= e^{-\mu p} \frac{(\mu p)^k}{k!}
\end{aligned}
$$

$\therefore \quad X \sim \text{Poisson}(\mu p).$