Math 5810 Exam 1, Solutions

1. a) $P(\text{exactly one}) = P(AB^cC^c \cup A^cBC^c \cup A^cB^cC) = P(AB^cC^c) + P(A^cBC^c) + P(A^cB^cC)$

$= P(A)P(B^c)P(C^c) + P(A^c)P(B)P(C^c) + P(A^c)P(B^c)P(C)$ $\quad$ [A, B, C independent]

$= \frac{1}{2}\cdot\frac{2}{3}\cdot\frac{4}{5} + \frac{1}{2}\cdot\frac{1}{3}\cdot\frac{4}{5} + \frac{1}{2}\cdot\frac{2}{3}\cdot\frac{1}{5} = \frac{8}{30} + \frac{4}{30} + \frac{2}{30} = \frac{14}{30} = \frac{7}{15}.$

b) $P(\text{at least one}) = P(A \cup B \cup C) = 1 - P((A \cup B \cup C)^c) = 1 - P(A^cB^cC^c) =$

$= 1 - P(A^c)P(B^c)P(C^c) = 1 - \frac{1}{2}\cdot\frac{2}{3}\cdot\frac{4}{5} = 1 - \frac{4}{15} = \frac{11}{15}.$

c) $$P(A \mid \qquad \text{one}) = \frac{P(A \cap \{\text{exactly one}\})}{P(\text{exactly one})} = \frac{P(AB^cC^c)}{P(\text{exactly one})} = \frac{\frac{1}{2}\cdot\frac{2}{3}\cdot\frac{4}{5}}{\frac{7}{15}} = \frac{4}{7}.$$

2. Let $H_i = \{\text{heads in } i^{th} \text{ toss}\}$ and $T_i = \{\text{tails in } i^{th} \text{ toss}\}$, $i = 1, 2, \ldots$.
Then $P_1 = 0$ (since at least two tosses are needed for two heads),

$P_2 = P(H_1H_2) = P(H_1)P(H_2) = \frac{1}{2}\cdot\frac{1}{2} = \frac{1}{4}$,

$P_3 = P(T_1H_2H_3) = P(T_1)P(H_2)P(H_3) = (\frac{1}{2})^3 = \frac{1}{8}$,

$P_4 = P(T_1T_2H_3H_4) + P(H_1T_2H_3H_4) = (\frac{1}{2})^4 + (\frac{1}{2})^4 = \frac{1}{8}$

$P_5 = P(T_1T_2T_3H_4H_5) + P(T_1H_2T_3H_4H_5) + P(H_1T_2T_3H_4H_5) = 3\cdot(\frac{1}{2})^5 = \frac{3}{32}.$

[In fact, for all r, $P_r = F_{r-1}/2^r$, where F_n denotes the n^{th} Fibonacci number!]

3. If $B_1, \ldots, B_n$ is a partition of B, then $AB_1, \ldots, AB_n$ is a partition of AB, so

$$P(AB) = P(AB_1) + P(AB_2) + \cdots + P(AB_n)$$
$$= P(A|B_1)P(B_1) + \cdots + P(A|B_n)P(B_n).$$

Divide by $P(B)$, and note that for $i = 1, \ldots, n$, $B_i \subseteq B$ and hence $B_i = B_iB$.
Thus,

$$P(A|B) = \frac{P(AB)}{P(B)} = P(A|B_1)\frac{P(B_1)}{P(B)} + \cdots + P(A|B_n)\frac{P(B_n)}{P(B)}$$
$$= P(A|B_1)\frac{P(B_1B)}{P(B)} + \cdots + P(A|B_n)\frac{P(B_nB)}{P(B)}$$
$$= P(A|B_1)P(B_1|B) + \cdots + P(A|B_n)P(B_n|B).$$

4. a) $P(\text{white}) = P(\text{white}|\text{Box 1})P(\text{Box 1}) + P(\text{white}|\text{Box 2})P(\text{Box 2})$

$= \frac{1}{3}\cdot\frac{1}{2} + \frac{3}{5}\cdot\frac{1}{2} = \frac{1}{6} + \frac{3}{10} = \frac{5}{30} + \frac{9}{30} = \frac{14}{30} = \frac{7}{15}$ (again!)

b) $$P(\text{Box 1}|\text{white}) = \frac{P(\text{white}|\text{Box 1})P(\text{Box 1})}{P(\text{white})} = \frac{\frac{1}{3}\cdot\frac{1}{2}}{\frac{7}{15}} = \frac{1}{6}\cdot\frac{15}{7} = \frac{5}{14}$$

c) $$P(\text{Box 1}|\text{black}) = \frac{P(\text{black}|\text{Box 1})P(\text{Box 1})}{P(\text{black})} = \frac{\frac{2}{3}\cdot\frac{1}{2}}{1-\frac{7}{15}} = \frac{1/3}{8/15} = \frac{5}{8}.$$

d) If the ball is white, guess box 2 (correct with probability $\frac{9}{14}$)
and if black, guess box 1 (correct with probability $\frac{5}{8}$). Then:

$$P(\text{guess correct}) = P(\text{guess correct}\mid\text{black})\,P(\text{black}) + P(\text{guess correct}\mid\text{white})\,P(\text{white})$$
$$= \frac{5}{8}\cdot\frac{8}{15} + \frac{9}{14}\cdot\frac{7}{15}$$
$$= \frac{1}{3} + \frac{9}{30} = \frac{19}{30}.$$

5. Each roll is an independent trial with 6 possible outcomes $(1,2,\dots,6)$, each of which has probability $p_i = \frac{1}{6}$ $(i=1,\dots,6)$. Let $N_1 = \#$ of ones, $N_2 = \#$ of twos, ..., $N_6 = \#$ of sixes. Then $(N_1,\dots,N_6) \sim \text{multinomial}(12; \frac{1}{6},\dots,\frac{1}{6})$, and the required probability is
$$P(N_1=2, N_2=2, \dots, N_6=2) = \frac{12!}{2!2!\cdots2!}\left(\frac{1}{6}\right)^2\left(\frac{1}{6}\right)^2\cdots\left(\frac{1}{6}\right)^2 = \frac{12!}{2^6}\left(\frac{1}{6}\right)^{12}.$$

6. Let X be the number of working devices in a box chosen at random. Then $X \sim \text{binomial}(n,p)$ with $n=50$ and $p=0.9$. Let $\mu = np = 45$, and $\sigma = \sqrt{npq} \doteq 2.12$. We want the largest k so that $P(X \geq k) \geq 0.9$. Using the normal approximation,
$$P(X \geq k) \approx 1 - \Phi\left(\frac{k-\frac{1}{2}-\mu}{\sigma}\right) = 1 - \Phi\left(\frac{k-45.5}{2.12}\right) = \Phi\left(\frac{45.5-k}{2.12}\right)$$
From the normal table, this last expression is ≥ 0.9 if and only if
$$\frac{45.5-k}{2.12} \geq 1.29 \Leftrightarrow 45.5-k \geq 2.73 \Leftrightarrow k \leq 42.76$$
So the largest integer k that satisfies the inequality is $\boxed{k=42}$.

7. a) There are 4 suits and 10 straight flushes per suit: $(A,2,3,4,5)$, $(2,3,4,5,6)$, ..., $(10,J,Q,K,A)$. Thus,
$$P(\text{straight flush}) = \frac{4\times 10}{\binom{52}{5}} \simeq 1.54\times 10^{-5} \text{ or } 0.0015\%$$

b) There are 13 ways to chose the "a" rank, and for each of these, $\binom{12}{3}$ ways to chose the remaining 3 ranks, whose order does not matter. Thus,
$$P(\text{a pair}) = 13 \times \binom{12}{3} \times \frac{\binom{4}{2}\binom{4}{1}\binom{4}{1}\binom{4}{1}}{\binom{52}{5}} \simeq 0.4226 \text{ or } 42.26\%$$
(See "list of poker hands" in Wikipedia)

8. a)

$Y_2 \backslash Y_1$	1	2	3	4
1	1/16	0	0	0
2	1/8	1/16	0	0
3	1/8	1/8	1/16	0
4	1/8	1/8	1/8	1/16

[e.g. $P(Y_1=1, Y_2=2) =$
$= P(X_1=1, X_2=2) + P(X_1=2, X_2=1)$
$= \frac{1}{16} + \frac{1}{16} = \frac{1}{8}$, etc.]

b) $\text{Range}(D) = \{0,1,2,3\}$.

c) $P(D=0) = P(Y_2 - Y_1 = 0) = P(Y_1=1, Y_2=1) + P(Y_1=2, Y_2=2) + P(Y_1=3, Y_2=3) + P(Y_1=4, Y_2=4) = \frac{4}{16} = \frac{1}{4}$

$P(D=1) = P(Y_2 - Y_1 = 1) = P(Y_1=1, Y_2=2) + P(Y_1=2, Y_2=3) + P(Y_1=3, Y_2=4) = \frac{3}{8}$.

$P(D=2) = P(Y_2 - Y_1 = 2) = P(Y_1=1, Y_2=3) + P(Y_1=2, Y_2=4) = \frac{2}{8} = \frac{1}{4}$.

$P(D=3) = P(Y_2 - Y_1 = 3) = P(Y_1=1, Y_2=4) = \frac{1}{8}$.

d) For $i = 2,3,4$:

$$P(Y_2 = i \mid D=1) = \text{[redacted]} \frac{P(Y_2 = i,\ D=1)}{P(D=1)} = \frac{P(Y_2 = i,\ Y_1 = i-1)}{P(D=1)} = \frac{1/8}{3/8} = \frac{1}{3}$$

so $(Y_2 \mid D=1) \sim$ uniform $\{2,3,4\}$.

9. a) If A_r happens for $r \geq 3$, the first two tosses can't both be heads, so either H_1^c or $H_1 H_2^c$ happens. Hence,

$$\left.\begin{aligned} P(A_r) &= P(A_r \cap H_1^c) + P(A_r \cap H_1 H_2^c) \\ &= P(A_r \mid H_1 H_2^c)\, P(H_1 H_2^c) + P(A_r \mid H_1^c)\, P(H_1^c) \\ &= \tfrac{1}{4} P(A_r \mid H_1 H_2^c) + \tfrac{1}{2} P(A_r \mid H_1^c). \end{aligned}\right] \quad (1)$$

b) If H_1^c happens, the first toss is tails and hence useless, and the game essentially "starts over" with the second toss, by independence of the tosses. Thus,

$$P(A_r \mid H_1^c) = P(A_{r-1}) = P_{r-1}. \qquad (2)$$

Likewise, if $H_1 H_2^c$ happens, then the first two tosses were useless and the game starts over from the third toss. Thus,

$$P(A_r \mid H_1 H_2^c) = P(A_{r-2}) = P_{r-2}. \qquad (3)$$

Substituting (2) and (3) into (1) gives $P_r = \frac{1}{2} P_{r-1} + \frac{1}{4} P_{r-2}$. (4)

Remark: Define $F_r := 2^r P_r$. Then (4) becomes (multiply both sides by 2^r):

$$F_r = F_{r-1} + F_{r-2},$$

the recursion for the Fibonacci numbers. Here, the sequence $\{F_r\}$ starts as $0, 1, 1, 2, 3, 5, 8, 13, \ldots$ rather than the usual $1, 1, 2, 3, 5, 8, 13, \ldots$.