Math 5810 - Exam 1 **October 9, 2013** **P. Allaart**

SHOW ALL YOUR WORK! NO WORK=NO CREDIT!!

Instructions:

1. Unless explicitly stated otherwise, you need not compute probabilities out to a decimal. Binomial coefficients, fractions, etc. may be left in the answer unless the problem tells you otherwise. However, obvious algebraic simplifications must be made (e.g. $\frac{1}{2} - \frac{1}{3} = \frac{1}{6}$).
2. Make sure your method can be understood. Clearly define any events and/or random variables that you use if they are not already defined in the problem.

1. *(4 pts. each)* Let A, B and C be independent events, with probabilities $\frac{1}{2}, \frac{1}{3}$ and $\frac{1}{5}$, respectively.

a) Find the probability that exactly one of the events occurs.

b) Find the probability that at least one of the events occurs.

c) Find the probability that A occurs, given that exactly one of the events occurs.

2. *(12 pts.)* A fair coin is tossed repeatedly until two heads in a row appear. Calculate the probability P_r that exactly r tosses are needed, for $r = 1, 2, \ldots, 5$.

3. *(10 pts.)* Prove that if $B_1, \ldots, B_n$ is a partition of B, then

$$\mathrm{P}(A|B) = \mathrm{P}(A|B_1)\mathrm{P}(B_1|B) + \cdots + \mathrm{P}(A|B_n)\mathrm{P}(B_n|B).$$

Show every step!

4. *(4 pts. each)* Consider two boxes: Box 1 contains 1 white ball and 2 black balls, and Box 2 contains 3 white balls and 2 black balls. A box is chosen at random, and a ball is drawn from that box at random.

a) Find the probability that the ball is white.

b) Find the probability that Box 1 was chosen, given that the ball is white.

c) Find the probability that Box 1 was chosen, given that the ball is black.

(*Problem 4, ctd.*) Suppose that after seeing the ball, you must guess which box it came from. Your strategy is to guess the box with the highest posterior probability given the color of the ball.

d) What is your (unconditional!) probability of guessing correctly?

5. *(9 pts.)* Twelve dice are rolled. Find the chance that each number from $\{1, 2, \ldots, 6\}$ appears exactly twice.

6. *(13 pts.)* A company produces electronic devices that work properly with probability 0.9, independently of each other. The devices are sold in boxes of 50 each. Find the largest k such that at least 90% of the boxes have the property that they contain k or more working devices.

7. *(6 pts. each)* A 5-card poker hand is dealt from a standard deck of 52 cards. Find the chance of being dealt:

   a) a straight flush (5 consecutive ranks of the same suit)

   b) a pair (ranks a,a,b,c,d, where a,b,c,d are all distinct)

8. Two FOUR-sided dice are rolled. Let X_1 and X_2 be the numbers that appear. Let $Y_1 = \min(X_1, X_2)$ and $Y_2 = \max(X_1, X_2)$, and define $D := Y_2 - Y_1$.

   a) *(4 pts.)* Construct the joint distribution table for Y_1 and Y_2

   b) *(2 pts.)* Give the *range* of D.

   c) *(4 pts.)* Find the *distribution* of D.

   d) *(6 pts.)* Find the conditional distribution of Y_2, given that $D = 1$.

9. **Extra credit!!** Consider the setting of problem 2. Let A_r be the event that exactly r tossed are needed ($r = 1, 2, \dots$), and let H_i be the event that the i-th toss lands heads ($i = 1, 2, \dots$).

   a) *(5 pts.)* Show that for $r \geq 3$,

$$\mathrm{P}(A_r) = \mathrm{P}(A_r | H_1 H_2^c)\mathrm{P}(H_1 H_2^c) + \mathrm{P}(A_r | H_1^c)\mathrm{P}(H_1^c).$$

   b) *(5 pts.)* Show that this implies that

$$\mathrm{P}_r = \tfrac{1}{2}\mathrm{P}_{r-1} + \tfrac{1}{4}\mathrm{P}_{r-2}$$

Explain your steps.