Math 5810, Review for final exam, Solutions

1. A = "player 1 gets two aces", B = "player 2 gets two aces"

$$P(AB) = P(A)P(B|A) = \frac{\binom{4}{2}\binom{48}{5}}{\binom{52}{7}} \cdot \frac{\binom{2}{2}\binom{43}{5}}{\binom{45}{7}}$$

2. F = "fair coin chosen", A = "4 heads"; $P(F) = P(F^c) = \frac{1}{2}$,

$P(A|F) = \binom{5}{4}\left(\frac{1}{2}\right)^4\left(\frac{1}{2}\right) = \frac{5}{32}$, $P(A|F^c) = \binom{5}{4}\left(\frac{2}{3}\right)^4\left(\frac{1}{3}\right) = \frac{80}{243}$. Then

$$P(A) = P(A|F)P(F) + P(A|F^c)P(F^c) = \frac{5}{32}\cdot\frac{1}{2} + \frac{80}{243}\cdot\frac{1}{2} = \frac{5}{64} + \frac{40}{243} = \frac{3775}{15552}$$

and

$$P(F|A) = \frac{P(A|F)P(F)}{P(A)} = \frac{5/64}{3775/15552} = \frac{243}{755} \approx 0.3219$$

3. (i) $\mu = \int_1^\infty x\cdot 3x^{-4}\,dx = 3\int_1^\infty x^{-3}\,dx = -\frac{3}{2}x^{-2}\Big|_1^\infty = \frac{3}{2}$

(ii) $F(x) = \int_1^x f(u)\,du = 1 - \int_x^\infty 3u^{-4}\,du = 1 - (-u^{-3})\Big|_x^\infty = 1 - x^{-3}$

$F(x) = \frac{1}{2} \Leftrightarrow x^{-3} = \frac{1}{2} \Leftrightarrow x^3 = 2 \Leftrightarrow x = \sqrt[3]{2}$. Thus $m = \sqrt[3]{2}$.

(iii) $E(X^2) = \int_1^\infty x^2\cdot 3x^{-4}\,dx = 3\int_1^\infty x^{-2}\,dx = -3x^{-1}\Big|_1^\infty = 3$, so $Var(X) = 3 - \left(\frac{3}{2}\right)^2 = \frac{3}{4}$ and $SD(X) = \frac{\sqrt{3}}{2} = \sigma$.

(iv) Note that $|\mu - m| = \left|\frac{3}{2} - \sqrt[3]{2}\right| = .240 < .866 = \frac{\sqrt{3}}{2} = \sigma$.

4. Let X denote a representative IQ score, so $E(X) = 100$.

a) $P(X > 130) \le \frac{100}{130} = \frac{10}{13} \approx 76.9\%$

b) $P(X > 130) \le P(|X - 100| > 30) \le \frac{Var(X)}{30^2} = \frac{10^2}{30^2} = \frac{1}{9} \approx 11.1\%$

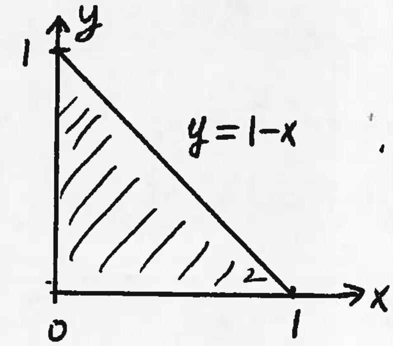
c) $P(X > 130) = 1 - \Phi\left(\frac{130-100}{10}\right) = 1 - \Phi(3) = 1 - .9987 = .0013 = 0.13\%$

5. a) Range(X) = Range(Y) = (0,1). For $0 < x < 1$,

$f_X(x) = \int f(x,y)\,dy = \int_0^{1-x} 6y\,dy = 3y^2\Big|_0^{1-x} = 3(1-x)^2$.

For $0 < y < 1$, $f_Y(y) = \int f(x,y)\,dx = \int_0^{1-y} 6y\,dy = 6y(1-y)$.

($X \sim beta(1,3)$, $Y \sim beta(2,2)$.)



b) For $0 < y < 1-x$,

$$f_Y(y|X=x) = \frac{f(x,y)}{f_X(x)} = \frac{6y}{3(1-x)^2} = \frac{2y}{(1-x)^2}$$

c) $E(Y|X=\frac{1}{2}) = \int_0^{1-\frac{1}{2}} y f_Y(y|X=\tfrac{1}{2})\,dy = \int_0^{1/2} y\cdot\frac{2y}{(\frac{1}{2})^2}\,dy = \int_0^{1/2} 8y^2\,dy$

$= \frac{8}{3}y^3\Big|_0^{1/2} = \frac{1}{3}.$

6. a) $N = I_A + I_B + I_C \Rightarrow E(N) = P(A) + P(B) + P(C) = \frac{1}{6} + \frac{1}{4} + \frac{1}{3} = \frac{3}{4}$.

b) $Var(N) = Var(I_A) + Var(I_B) + Var(I_C) = P(A)(1-P(A)) + P(B)(1-P(B)) +$
$+ P(C)(1-P(C)) = \frac{1}{6}\cdot\frac{5}{6} + \frac{1}{4}\cdot\frac{3}{4} + \frac{1}{3}\cdot\frac{2}{3} = \frac{5}{36} + \frac{3}{16} + \frac{2}{9} = \frac{79}{144}$

c) $P(N=3) = P(A) = \frac{1}{6}$, $P(N=2) = P(B\setminus A) = \frac{1}{4} - \frac{1}{6} = \frac{1}{12}$, $P(N=1) =$
$= P(C\setminus B) = \frac{1}{3} - \frac{1}{4} = \frac{1}{12} \Rightarrow E(N^2) = 1^2\cdot P(N=1) + 2^2\cdot P(N=2) + 3^2\cdot P(N=3)$
$= 1\cdot\frac{1}{12} + 4\cdot\frac{1}{12} + 9\cdot\frac{1}{6} = \frac{5}{12} + \frac{9}{6} = \frac{23}{12} \Rightarrow Var(N) = \frac{23}{12} - \left(\frac{3}{4}\right)^2 = \frac{65}{48}$

7. a) $N \sim$ Poisson (80), so $P(N=k) = e^{-80}\frac{80^k}{k!}$, $k = 0, 1, 2, \ldots$.

b) $E(N) = Var(N) = 80$

c) $P(N > 100) \approx 1 - \Phi\left(\frac{100 + \frac{1}{2} - E(N)}{SD(N)}\right) = 1 - \Phi\left(\frac{100.5 - 80}{\sqrt{80}}\right)$
$\approx 1 - \Phi(2.29) = 1 - .9890 = .0110$.
Normal approximation is justified because $N = N_1 + \ldots + N_{20}$, where $N_1, \ldots, N_{20}$ are independent and each Poisson (4) distributed. Since $n = 20$ is rather small, continuity correction is required!

8. a)

Let X be the position of the accident, then $X \sim$ uniform $(-30, 70)$, and $T = \frac{|X|}{60} \times 60 = |X|$ minutes, so Range $(T) = (0, 70)$.

b) $F_T(t) = P(T \le t) = P(|X| \le t) = P(-t \le X \le t) = \begin{cases} \frac{2t}{100} = \frac{t}{50}, & \text{if } 0 < t < 30 \\ \frac{t-(-30)}{100} = \frac{t+30}{100} & \text{if } 30 \le t < 70. \end{cases}$

9. Let $N =$ total number of misprints, $X = \#$ found. Then $N \sim$ Poisson(3) and $(X|N=n) \sim$ binomial $(n, 0.9)$. So for $k = 0, 1, 2, \ldots$,

$$P(X=k) = \sum_{n=k}^{\infty} P(X=k|N=n)P(N=n) = \sum_{n=k}^{\infty}\binom{n}{k}p^k(1-p)^{n-k}e^{-\mu}\frac{\mu^n}{n!} \quad (p=0.9,\ \mu=3)$$
$$= p^k e^{-\mu}\cdot\frac{1}{k!}\sum_{n=k}^{\infty}\frac{(1-p)^{n-k}}{(n-k)!}\mu^n = \mu^k p^k e^{-\mu}\cdot\frac{1}{k!}\sum_{n=k}^{\infty}\frac{[\mu(1-p)]^{n-k}}{(n-k)!}$$
$$= e^{-\mu}\cdot\frac{(\mu p)^k}{k!}e^{\mu(1-p)} = e^{-\mu p}\frac{(\mu p)^k}{k!} \Rightarrow X \sim \text{Poisson}(\mu p = 2.7)$$

So $P(X \ge 2) = 1 - P(X \le 1) = 1 - e^{-2.7} - 2.7e^{-2.7} \approx .7513$

10. Range $(Z) = (0,1)$. For $0<z<1$,

$$F_Z(z) = P(Z \le z) = P(X^2+Y^2 \le z) = P((X,Y) \in A) \quad (A = \text{disk w. radius } \sqrt{z} \text{ centered at } (0,0))$$

$$= \frac{\text{area}(A)}{\pi} = \frac{\pi(\sqrt{z})^2}{\pi} = z \Rightarrow f_Z(z) = 1, \; 0<z<1.$$

11. Note $S_4 = S_1 + X_2 + X_3 + X_4$ where S_1, X_2, X_3, X_4 are indep. exp (λ).
So

$$Cov(S_1, S_4) = Cov(S_1, S_1 + X_2 + X_3 + X_4) = Cov(S_1, S_1) + Cov(S_1, X_2+X_3+X_4)$$

$$= Var(S_1) + 0 = \frac{1}{\lambda^2}$$

Also

$$Var(S_1) = \frac{1}{\lambda^2}, \quad Var(S_4) = \frac{4}{\lambda^2} \Rightarrow SD(S_1) = \frac{1}{\lambda}, \quad SD(S_4) = \frac{2}{\lambda}$$

$$\therefore Corr(S_1, S_4) = \frac{Cov(S_1,S_4)}{SD(S_1)SD(S_4)} = \frac{1}{2}.$$

12. Three lengths can be the sides of a triangle if none of them exceeds the sum of the other two. Assume the stick has length 1. Let X and Y be the break points, so X and Y are independent uniform $(0,1)$. The above condition is now equivalent to each of the three pieces having length at most $\frac{1}{2}$. This is the case if and only if

(i) $X \le Y$ and $X \le \frac{1}{2}$, $Y - X \le \frac{1}{2}$, $1 - Y \le \frac{1}{2}$, so $X \le \frac{1}{2}$, $Y \le X + \frac{1}{2}$, $Y \ge \frac{1}{2}$;
or
(ii) $X > Y$ and $Y \le \frac{1}{2}$, $X - Y \le \frac{1}{2}$, $1 - X \le \frac{1}{2}$ so $Y \le \frac{1}{2}$, $X \le Y + \frac{1}{2}$, $X \ge \frac{1}{2}$.

Thus,

$$P(\text{can make a triangle}) = P((X,Y) \in A \cup B)$$

$$= \text{area}(A \cup B) = \frac{1}{4}.$$