Math 460, Review for Exam 3, Solutions

1. Let T be the number of times the die is rolled until either Amy or Beth wins. Note Amy wins if T is odd, Beth wins if T is even. Let $A_i = \{5 \text{ or } 6 \text{ on the } i^{th} \text{ roll}\}$, $B_i = \{4, 5 \text{ or } 6 \text{ on the } i^{th} \text{ roll}\}$. Then $P(A_i) = \frac{1}{3}$, $P(B_i) = \frac{1}{2}$, so
$$P(T = 2n-1) = P(A_1^c B_2^c A_3^c B_4^c \cdots A_{2n-3}^c B_{2n-2}^c A_{2n-1}) = \left(\tfrac{2}{3}\cdot\tfrac{1}{2}\right)^{n-1}\cdot\tfrac{1}{3} = \tfrac{1}{3}\left(\tfrac{1}{3}\right)^{n-1}$$
$$= \left(\tfrac{1}{3}\right)^n, \quad \text{for } n = 1, 2, \ldots,$$
and so
$$P(\text{Amy wins}) = P(T \text{ is odd}) = \sum_{n=1}^{\infty} P(T = 2n-1) = \sum_{n=1}^{\infty} \left(\tfrac{1}{3}\right)^n = \frac{1/3}{1-\frac{1}{3}} = \frac{1}{2}.$$

2. a) $P(\text{Sum of dice} \geq 10) = P(\{(4,6), (5,5), (5,6), (6,4), (6,5), (6,6)\}) = \frac{6}{36} = \frac{1}{6}$
Let T be the number of rolls, then $T \sim \text{geometric}\left(\frac{1}{6}\right)$, so $P(T=5) = \left(\frac{5}{6}\right)^4 \cdot \frac{1}{6}$.
b) $E(T) = \frac{1}{1/6} = 6$.

3. $\text{Range}(Y) = (0,1)$. For $0 < y < 1$, $F_Y(y) = P(Y \leq y) = P(\max\{X_1, X_2, X_3\} \leq y)$
$= P(X_1 \leq y, X_2 \leq y, X_3 \leq y) = P(X_1 \leq y)P(X_2 \leq y)P(X_3 \leq y) = y^3$, and $f_Y(y) = F_Y'(y) = 3y^2$.

4. Let $X_1, \ldots, X_{30}$ denote the amounts of the individual tips. Assume $X_1, \ldots, X_n$ are <u>independent</u> and have the <u>same distribution</u>. Let $S_{30} = X_1 + \cdots + X_{30}$. By the Central Limit Theorem, S_{30} is approximately normal, and $P(S_{30} > 200) \approx 1 - \Phi\left(\frac{170-150}{1.5\sqrt{30}}\right) \approx 1 - \Phi(2.43)$
$= 1 - .9925 = .0075$. (<u>Note</u>: continuity correction makes little sense here as tips generally aren't whole dollar amounts.)

5. $1 = \int_{-\infty}^{\infty} f(x)\,dx = \int_0^1 c\sqrt{x}\,dx = c\int_0^1 x^{1/2}\,dx = c\cdot\frac{2}{3}x^{3/2}\Big|_0^1 = \frac{2}{3}c \quad \Rightarrow c = \boxed{\frac{3}{2}}.$

6. a) Let N be the number of raisins in the cookie. Assume N has a Poisson(4) distribution. Then $P(N \geq 1) = 1 - P(N=0) = 1 - e^{-4} = 0.9817$
b) Let N_1 be the number of raisins in the first cookie, N_2 the number of raisins in the second. We may assume N_1, N_2 are independent. Then $N_1 + N_2 \sim \text{Poisson}(4+4=8)$, so $P(N_1 + N_2 = 8) = e^{-8}\cdot\frac{8^8}{8!} = 0.1396$.
c) Must find μ so that $P(N \geq 1) = 1 - e^{-\mu} = 0.95 \Rightarrow e^{-\mu} = 0.05 \Rightarrow \mu = -\ln(0.05)$
$= \ln 20 = 2.9957 \approx 3.0$. So she can reduce the average number of raisins per cookie by one full raisin.

7. $F_X(x) = x/4$ for $0 < x < 4$. $\text{Range}(Y) = (0,2)$. For $0 < y < 2$, $F_Y(y) = P(Y \leq y)$
$= P(\sqrt{X} \leq y) = P(X \leq y^2) = F_X(y^2) = y^2/4$, and $f_Y(y) = F_Y'(y) = y/2$.

8. <u>Method 1</u>: $E(e^T) = \int_{-\infty}^{\infty} e^t f(t)\,dt = \int_0^{\infty} e^t \cdot \lambda e^{-\lambda t}\,dt = \int_0^{\infty} \lambda e^{-(\lambda-1)t}\,dt$
$= \frac{\lambda}{\lambda-1}\int_0^{\infty}(\lambda-1)e^{-(\lambda-1)t}\,dt = \frac{\lambda}{\lambda-1}$, since $(\lambda-1)e^{-(\lambda-1)t}$ is the density of an exponential $(\lambda-1)$ distribution.

<u>Method 2</u>: $E(e^T) = \int_0^{\infty} P(e^T > x)\,dx = \int_0^1 1\,dx + \int_1^{\infty} P(T > \ln x)\,dx$ $(e^T \geq 1)$
$= 1 + \int_1^{\infty} e^{-\lambda \ln x}\,dx = 1 + \int_1^{\infty} (e^{\ln x})^{-\lambda}\,dx = 1 + \int_1^{\infty} x^{-\lambda}\,dx = 1 + \left.\frac{x^{1-\lambda}}{1-\lambda}\right|_1^{\infty}$
$= 1 + \left(0 - \frac{1}{1-\lambda}\right) = 1 - \frac{1}{1-\lambda} = \frac{-\lambda}{1-\lambda} = \frac{\lambda}{\lambda-1}$.

(Method 1 is standard; Method 2 is ad hoc.)