Math 4610, Review for Exam 2, Solutions

1. Sampling without replacement!

a) There are $\binom{52}{13}$ total possible hands, all equally likely. There are $\binom{4}{2}$ ways to choose 2 aces out of 4, and $\binom{48}{11}$ ways to choose 11 non-aces out of 48. Thus,

$$P(\text{two aces}) = \binom{4}{2}\binom{48}{11}\Big/\binom{52}{13}.$$

b) There are $\binom{4}{2}$ ways to choose 2 aces, $\binom{4}{3}$ ways to choose 3 kings, and $\binom{44}{8}$ ways to choose 8 other cards. Thus, $P(\text{two aces, three kings}) = \binom{4}{2}\binom{4}{3}\binom{44}{8}\Big/\binom{52}{13}$.

2. Each roll is an independent trial with 6 possible outcomes $(1,2,\dots,6)$, each of which has probability $p_i = \frac{1}{6}$ $(i=1,\dots,6)$. Let $N_1 = \#$ of ones, $N_2 = \#$ of twos, ..., $N_6 = \#$ of sixes. Then $(N_1,\dots,N_6) \sim$ multinomial $(12; \frac{1}{6},\dots,\frac{1}{6})$, and the required probability is

$$P(N_1=2, N_2=2, \dots N_6=2) = \frac{12!}{2!2!\cdots2!}\left(\frac{1}{6}\right)^2\left(\frac{1}{6}\right)^2\cdots\left(\frac{1}{6}\right)^2 = \frac{12!}{2^6}\left(\frac{1}{6}\right)^{12}.$$

3. $E(XY) = E(X)E(Y) = \mu^2$, and $E[(XY)^2] = E[X^2Y^2] = E(X^2)E(Y^2)$
$= \{Var(X) + (E(X))^2\}\cdot\{Var(Y) + (E(Y))^2\} = (\sigma^2+\mu^2)^2$, so $Var(XY) = E[(XY)^2] - [E(XY)]^2$
$= (\sigma^2+\mu^2)^2 - (\mu^2)^2 = (\sigma^4 + 2\sigma^2\mu^2 + \mu^4) - \mu^4 = \sigma^4 + 2\sigma^2\mu^2 = \sigma^2(\sigma^2+2\mu^2)$.

4. a)

$X_2 \backslash X_1$	1	2	3	4
1	1/16	1/16	1/16	1/16
2	1/16	1/16	1/16	1/16
3	1/16	1/16	1/16	1/16
4	1/16	1/16	1/16	1/16

b)

$Y_2 \backslash Y_1$	1	2	3	4
1	1/16	0	0	0
2	1/8	1/16	0	0
3	1/8	1/8	1/16	0
4	1/8	1/8	1/8	1/16

[e.g. $P(Y_1=1, Y_2=2) = P(X_1=1, X_2=2) + P(X_1=2, X_2=1) = \frac{1}{16}+\frac{1}{16} = \frac{1}{8}$, etc.]

c) Range of D: $\{0,1,2,3\}$.

d) $P(D=0) = P(Y_2-Y_1=0) = P(Y_1=1,Y_2=1) + P(Y_1=2,Y_2=2) + P(Y_1=3,Y_2=3) + P(Y_1=4,Y_2=4) = \frac{4}{16} = \frac{1}{4}$.

$P(D=1) = P(Y_2-Y_1=1) = P(Y_1=1,Y_2=2) + P(Y_1=2,Y_2=3) + P(Y_1=3,Y_2=4) = \frac{3}{8}$.

$P(D=2) = P(Y_2-Y_1=2) = P(Y_1=1,Y_2=3) + P(Y_1=2,Y_2=4) = \frac{2}{8} = \frac{1}{4}$.

$P(D=3) = P(Y_2-Y_1=3) = P(Y_1=1,Y_2=4) = \frac{1}{8}$.

e) $E(D) = 0\cdot P(D=0) + 1\cdot P(D=1) + 2\cdot P(D=2) + 3\cdot P(D=3)$
$= 0\cdot\frac{1}{4} + 1\cdot\frac{3}{8} + 2\cdot\frac{1}{4} + 3\cdot\frac{1}{8} = \frac{5}{4}$.

5. a) Let X be the income of a randomly chosen family. By Markov's inequality,

$$P(X>100{,}000) \le \frac{E(X)}{100{,}000} = \frac{60{,}000}{100{,}000} = 0.6, \text{ or } 60\% \text{ of families.}$$

b) By Chebychev's inequality,

$$P(X>100{,}000) \le P(X<20{,}000 \text{ or } X>100{,}000) = P(|X-E(X)| \ge 40{,}000)$$
$$= P(|X-E(X)| \ge 4\,SD(X)) \le \frac{1}{4^2} = \frac{1}{16}, \text{ or } 6.25\% \text{ of families.}$$

6. a) The draws are independent trials, but with different success probabilities. Hence, X does not have a binomial distribution.

b) Let $A_i = \{\text{ball from } i^{th} \text{ urn is red}\}$, $i=1,2,3$. Then $X = I_{A_1} + I_{A_2} + I_{A_3}$, so

$$E(X) = E(I_{A_1}) + E(I_{A_2}) + E(I_{A_3}) = P(A_1) + P(A_2) + P(A_3) = \frac{1}{3} + \frac{3}{4} + \frac{1}{2} = \frac{19}{12}.$$

7. Let X be the number of working devices in a box chosen at random. Then $X \sim$ binomial (n,p) with $n=50$ and $p=0.9$. Let $\mu = np = 45$, and $\sigma = \sqrt{npq} \doteq 2.12$.
We want the largest k so that $P(X \geq k) \geq 0.9$. Using the normal approximation,

$$P(X \geq k) \approx 1 - \Phi\left(\frac{k - \frac{1}{2} - \mu}{\sigma}\right) = 1 - \Phi\left(\frac{k - 45.5}{2.12}\right) = \Phi\left(\frac{45.5 - k}{2.12}\right)$$

From the normal table, this last expression is ≥ 0.9 if and only if

$$\frac{45.5 - k}{2.12} \geq 1.29 \Leftrightarrow 45.5 - k \geq 2.73 \Leftrightarrow k \leq 42.76$$

So the largest integer k that satisfies the inequality is $\boxed{k = 42}$

8. Let $p = P(\text{a given seed does not germinate}) = 0.01$. Let $N = \#$ of seeds that don't germinate. Then $N \sim Bin(n,p)$ where $n = 50$, $p = 0.01$ so $\mu = np = 0.5$. Thus,

$$P(\geq 3 \text{ seeds don't germinate}) = P(N \geq 3) = 1 - P(N \leq 2) \approx 1 - e^{-\mu}\left(1 + \mu + \frac{\mu^2}{2}\right) = 0.0144.$$