Math 4610, Review for final exam, Solutions

1. $A =$ "player 1 gets two aces", $B =$ "player 2 gets two aces"

$$P(AB) = P(A)P(B|A) = \frac{\binom{4}{2}\binom{48}{5}}{\binom{52}{7}} \cdot \frac{\binom{2}{2}\binom{43}{5}}{\binom{45}{7}}$$

2. $F =$ "fair coin chosen", $A =$ "4 heads"; $P(F) = P(F^c) = \frac{1}{2}$,

$P(A|F) = \binom{5}{4}\left(\frac{1}{2}\right)^4\left(\frac{1}{2}\right) = \frac{5}{32}$, $P(A|F^c) = \binom{5}{4}\left(\frac{2}{3}\right)^4\left(\frac{1}{3}\right) = \frac{80}{243}$. Then

$$P(A) = P(A|F)P(F) + P(A|F^c)P(F^c) = \frac{5}{32}\cdot\frac{1}{2} + \frac{80}{243}\cdot\frac{1}{2} = \frac{5}{64} + \frac{40}{243} = \frac{3775}{15552}$$

and

$$P(F|A) = \frac{P(A|F)P(F)}{P(A)} = \frac{5/64}{3775/15552} = \frac{243}{755} \approx 0.3219$$

3. (i) $\mu = \int_1^\infty x\cdot 3x^{-4}\,dx = 3\int_1^\infty x^{-3}\,dx = -\frac{3}{2}x^{-2}\Big|_1^\infty = \frac{3}{2}$

(ii) $F(x) = \int_1^x f(u)\,du = 1 - \int_x^\infty 3u^{-4}\,du = 1 - \left(-u^{-3}\right)\Big|_x^\infty = 1 - x^{-3}$

$F(x) = \frac{1}{2} \Leftrightarrow x^{-3} = \frac{1}{2} \Leftrightarrow x^3 = 2 \Leftrightarrow x = \sqrt[3]{2}$. Thus $m = \sqrt[3]{2}$.

(iii) $E(X^2) = \int_1^\infty x^2\cdot 3x^{-4}\,dx = 3\int_1^\infty x^{-2}\,dx = -3x^{-1}\Big|_1^\infty = 3$, so $\mathrm{Var}(X) = 3 - \left(\frac{3}{2}\right)^2 = \frac{3}{4}$

and $SD(X) = \frac{\sqrt{3}}{2} = \sigma$.

(iv) Note that $|\mu - m| = \left|\frac{3}{2} - \sqrt[3]{2}\right| = .240 < .866 = \frac{\sqrt{3}}{2} = \sigma$.

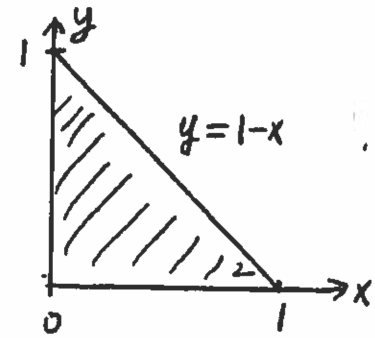
4. Let X denote a representative IQ score, so $E(X) = 100$.

a) $P(X > 130) \le \frac{100}{130} = \frac{10}{13} \approx 76.9\%$

b) $P(X > 130) \le P(|X - 100| > 30) \le \frac{\mathrm{Var}(X)}{30^2} = \frac{10^2}{30^2} = \frac{1}{9} \approx 11.1\%$

c) $P(X > 130) = 1 - \Phi\left(\frac{130-100}{10}\right) = 1 - \Phi(3) = 1 - .9987 = .0013 = 0.13\%$

5. a) $\mathrm{Range}(X) = \mathrm{Range}(Y) = (0,1)$. For $0 < x < 1$,



$$f_X(x) = \int f(x,y)\,dy = \int_0^{1-x} 6y\,dy = 3y^2\Big|_0^{1-x} = 3(1-x)^2.$$

For $0 < y < 1$, $f_Y(y) = \int f(x,y)\,dx = \int_0^{1-y} 6y\,dy = 6y(1-y)$.

($X \sim$ beta $(1,3)$, $Y \sim$ beta $(2,2)$.)

b) For $0 < y < 1-x$,

$$f_Y(y|X=x) = \frac{f(x,y)}{f_X(x)} = \frac{6y}{3(1-x)^2} = \frac{2y}{(1-x)^2}$$

c) $E(Y|X=\frac{1}{2}) = \int_0^{1-\frac{1}{2}} y\, f_Y(y|X=\frac{1}{2})\,dy = \int_0^{1/2} y\cdot\frac{2y}{\left(\frac{1}{2}\right)^2}\,dy = \int_0^{1/2} 8y^2\,dy$

$$= \frac{8}{3}y^3\Big|_0^{1/2} = \frac{1}{3}.$$

6. a) $N = I_A + I_B + I_C \Rightarrow E(N) = P(A) + P(B) + P(C) = \frac{1}{6} + \frac{1}{4} + \frac{1}{3} = \frac{3}{4}$.

b) $\mathrm{Var}(N) = \mathrm{Var}(I_A) + \mathrm{Var}(I_B) + \mathrm{Var}(I_C) = P(A)(1-P(A)) + P(B)(1-P(B)) + P(C)(1-P(C)) = \frac{1}{6}\cdot\frac{5}{6} + \frac{1}{4}\cdot\frac{3}{4} + \frac{1}{3}\cdot\frac{2}{3} = \frac{5}{36} + \frac{3}{16} + \frac{2}{9} = \frac{79}{144}$

c) $P(N=3) = P(A) = \frac{1}{6}$, $P(N=2) = P(B \setminus A) = \frac{1}{4} - \frac{1}{6} = \frac{1}{12}$, $P(N=1) = P(C \setminus B) = \frac{1}{3} - \frac{1}{4} = \frac{1}{12} \Rightarrow E(N^2) = 1^2 \cdot P(N=1) + 2^2 \cdot P(N=2) + 3^2 \cdot P(N=3)$
$= 1 \cdot \frac{1}{12} + 4 \cdot \frac{1}{12} + 9 \cdot \frac{1}{6} = \frac{5}{12} + \frac{9}{6} = \frac{23}{12} \Rightarrow \mathrm{Var}(N) = \frac{23}{12} - \left(\frac{3}{4}\right)^2 = \frac{65}{48}$

7. a) $N \sim \text{Poisson}(80)$, so $P(N=k) = e^{-80} \frac{80^k}{k!}$, $k = 0, 1, 2, \ldots$

b) $E(N) = \mathrm{Var}(N) = 80$

c) $P(N > 100) \approx 1 - \Phi\left(\frac{100 + \frac{1}{2} - E(N)}{SD(N)}\right) = 1 - \Phi\left(\frac{100.5 - 80}{\sqrt{80}}\right)$
$= 1 - \Phi(2.29) = 1 - .9890 = .0110$.

Normal approximation is justified because $N = N_1 + \ldots + N_{20}$, where $N_1, \ldots, N_{20}$ are independent and each Poisson (4) distributed. Since $n = 20$ is rather small, continuity correction is required!

8. a)

Let X be the position of the accident, then $X \sim \text{uniform}(-30, 70)$, and $T = \frac{|X|}{60} \times 60 = |X|$ minutes, so $\text{Range}(T) = (0, 70)$.

b) $F_T(t) = P(T \le t) = P(|X| \le t) = P(-t \le X \le t) = \begin{cases} \frac{2t}{100} = \frac{t}{50}, & \text{if } 0 < t < 30 \\ \frac{t-(-30)}{100} = \frac{t+30}{100} & \text{if } 30 \le t < 70. \end{cases}$

9. Let N = total number of misprints, X = # found. Then $N \sim \text{Poisson}(3)$ and $(X \mid N=n) \sim \text{binomial}(n, 0.9)$. So for $k = 0, 1, 2, \ldots$,

$$P(X=k) = \sum_{n=k}^{\infty} P(X=k \mid N=n) P(N=n) = \sum_{n=k}^{\infty} \binom{n}{k} p^k (1-p)^{n-k} e^{-\mu} \frac{\mu^n}{n!} \quad (p = 0.9, \mu = 3).$$

$$= p^k e^{-\mu} \cdot \frac{1}{k!} \sum_{n=k}^{\infty} \frac{(1-p)^{n-k}}{(n-k)!} \mu^n = \mu^k p^k e^{-\mu} \cdot \frac{1}{k!} \sum_{n=k}^{\infty} \frac{[\mu(1-p)]^{n-k}}{(n-k)!}$$

$$= e^{-\mu} \cdot \frac{(\mu p)^k}{k!} e^{\mu(1-p)} = e^{-\mu p} \frac{(\mu p)^k}{k!} \Rightarrow X \sim \text{Poisson}(\mu p = 2.7)$$

So $P(X \ge 2) = 1 - P(X \le 1) = 1 - e^{-2.7} - 2.7 e^{-2.7} \approx .7513$

10. Range $(Z) = (0,1)$. For $0<z<1$,

$F_Z(z) = P(Z \le z) = P(X^2+Y^2 \le z) = P((X,Y) \in A)$ (A = disk w. radius $\sqrt{z}$ centered at $(0,0)$)

$= \frac{\text{area}(A)}{\pi} = \frac{\pi(\sqrt{z})^2}{\pi} = z \Rightarrow f_Z(z) = 1, \ 0<z<1.$

11. Let Z = # of heads in tosses 101–500. Then $Y = X+Z$, and X and Z are independent. So $Cov(X,Y) = Cov(X, X+Z)$

$= Cov(X,X) + Cov(X,Z) = Var(X) + 0 = Var(X)$.

Now $X \sim$ binomial $(100, \frac{1}{2})$ and $Y \sim$ binomial $(500, \frac{1}{2})$, so

$SD(X) = \sqrt{100 \cdot \frac{1}{2} \cdot \frac{1}{2}} = 5$, $SD(Y) = \sqrt{500 \cdot \frac{1}{2} \cdot \frac{1}{2}} = \sqrt{125}$, $Var(X) = 5^2 = 25$.

$\therefore Corr(X,Y) = \frac{Cov(X,Y)}{SD(X)SD(Y)} = \frac{Var(X)}{SD(X)SD(Y)} = \frac{SD(X)}{SD(Y)} = \frac{5}{\sqrt{125}}$

$= \frac{1}{\sqrt{5}} \approx 0.4472$

12. For $x>0$, $f_X(x) = \int_0^x \lambda^2 e^{-\lambda x}\,dy = \lambda^2 x e^{-\lambda x} \Rightarrow X \sim$ gamma $(2,\lambda)$

$\Rightarrow E(X) = \frac{2}{\lambda}$.

For $y>0$, $f_Y(y) = \int_y^\infty \lambda^2 e^{-\lambda x}\,dx = \lambda e^{-\lambda y} \Rightarrow Y \sim$ exponential (λ)

$\Rightarrow E(Y) = \frac{1}{\lambda}$.

Next,

$$E(XY) = \int_0^\infty \int_0^x xy\, f(x,y)\,dy\,dx = \int_0^\infty \int_0^x xy\,\lambda^2 e^{-\lambda x}\,dy\,dx$$

$$= \frac{\lambda^2}{2} \int_0^\infty x^3 e^{-\lambda x}\,dx$$ (integrate by parts. $u = x^3$, $dv = e^{-\lambda x}dx$, $du = 3x^2 dx$, $v = -\frac{1}{\lambda}e^{-\lambda x}$)

$$= \frac{\lambda^2}{2}\left(\underbrace{-\frac{x^3}{\lambda}e^{-\lambda x}\Big|_0^\infty}_{0} + \int_0^\infty \frac{1}{\lambda}\cdot 3x^2 e^{-\lambda x}\,dx\right)$$

$$= \frac{3}{2}\int_0^\infty x^2 \lambda e^{-\lambda x}\,dx = \frac{3}{2}E(Y^2) = \frac{3}{2}\cdot\frac{2}{\lambda^2} = \frac{3}{\lambda^2}.$$

$$\therefore Cov(X,Y) = E(XY) - E(X)E(Y) = \frac{3}{\lambda^2} - \frac{2}{\lambda}\cdot\frac{1}{\lambda} = \frac{1}{\lambda^2}.$$