Answer Key, Hw. # 8.

§3.1 ① a)

x	0	1	2	3
$P(X=x)$	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{8}$

b)

y	0	1	2
$P(\lvert X-1\rvert=y)$	$\frac{3}{8}$	$\frac{4}{8}$	$\frac{1}{8}$

(e.g. $P(|X-1|=1) = P(X=0) + P(X=2) = \frac{1}{8} + \frac{3}{8} = \frac{4}{8}$ $(=\frac{1}{2})$).

⑧ a) Let X = # of cards until first ace appears. Range of $X = \{1,2,3,4\}$.

$P(X=1) = \frac{2}{5} = \frac{4}{10}$, $P(X=2) = \frac{3}{5} \times \frac{2}{4} = \frac{3}{10}$, $P(X=3) = \frac{3}{5} \times \frac{2}{4} \times \frac{2}{3} = \frac{2}{10}$,

$P(X=4) = \frac{3}{5} \times \frac{2}{4} \times \frac{1}{3} \times 1 = \frac{1}{10}$, $P(X=5) = 0$.

b) Let Y = # of cards until second ace appears. Range of $Y = \{2,3,4,5\}$.

For $k \in \{2,3,4,5\}$, the event $\{Y=k\}$ happens exactly when the first $k-1$ cards contain 1 ace and $k-2$ kings, and the k-th card is an ace. Thus,

$$P(Y=k) = \frac{\binom{2}{1}\binom{3}{k-2}}{\binom{5}{k-1}} \cdot \frac{1}{5-(k-1)} = \frac{2\binom{3}{k-2}}{\binom{5}{k-1}} \cdot \frac{1}{6-k}, \quad k=2,3,4,5.$$

$\therefore$ $P(Y=2) = \frac{1}{10}$, $P(Y=3) = \frac{2}{10}$, $P(Y=4) = \frac{3}{10}$, $P(Y=5) = \frac{4}{10}$: probabilities from (a) in reverse

c) When dealing from the bottom, the distributions of X and Y remain the same, but the first ace becomes the second ace and vice versa. Hence, $Y \sim 6-X$.

§3.1 ⑥ a)

$Y \backslash X$	0	1	2
0	$\frac{1}{8}$	$\frac{1}{8}$	0
1	$\frac{1}{8}$	$\frac{2}{8}$	$\frac{1}{8}$
2	0	$\frac{1}{8}$	$\frac{1}{8}$

– e.g. $P(X=1, Y=1) = P(\text{"hth" or "tht"}) = \frac{1}{8} + \frac{1}{8} = \frac{2}{8}$

b) Dependent, e.g. $P(X=2, Y=0) \neq P(X=2)P(Y=0)$

c) Let $S = X+Y$. Then for instance,

$P(S=1) = P(X=0, Y=1) + P(X=1, Y=0) = \frac{1}{8} + \frac{1}{8} = \frac{1}{4}$.

Distribution of S:

s	0	1	2	3	4
$P(S=s)$	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{8}$

⑫ a) $N_i \sim \text{binomial}(n, p_i)$ b) $N_i + N_j \sim \text{binomial}(n, p_i + p_j)$

c) $(N_i, N_j, n-N_i-N_j) \sim \text{multinomial}(n; p_i, p_j, 1-p_i-p_j)$.

(15) For all $i,j \in \{1,2,\dots,n\}$, $P(X=i,Y=j) = P(X=i)P(Y=j) = \frac{1}{n}\cdot\frac{1}{n} = \frac{1}{n^2}$.

a) $P(X=Y) = \sum_{i=1}^{n} P(X=i,Y=i) = \sum_{i=1}^{n} \frac{1}{n^2} = n\cdot\frac{1}{n^2} = \frac{1}{n}$.

b) $P(X<Y) = \sum_{i<j} P(X=i,Y=j) = \frac{1}{n^2}\cdot \#\{(i,j): i<j,\ i,j\in\{1,2,\dots,n\}\}$

$= \frac{1}{n^2}\cdot\left[(n-1)+(n-2)+\dots+2+1\right] = \frac{1}{n^2}\cdot\frac{n(n-1)}{2} = \frac{n-1}{2n}$

c) By symmetry, $P(X>Y) = P(X<Y) = \frac{n-1}{2n}$.

[Alternative to b) and c): since $P(X=Y)+P(X<Y)+P(X>Y)=1$ and $P(X<Y)=P(X>Y)$ by symmetry, it follows that $P(X<Y) = \frac{1}{2}\left[1-P(X=Y)\right] = \frac{n-1}{2n}$.]

d) $P(\max(X,Y)=k) = P(X=k, Y\le k) + P(X<k, Y=k) = P(X=k)P(Y\le k) + P(X<k)P(Y=k)$

$= \frac{1}{n}\cdot\frac{k}{n} + \frac{k-1}{n}\cdot\frac{1}{n} = \frac{2k-1}{n^2}$

e) Similarly, $P(\min(X,Y)=k) = \frac{2(n+1-k)-1}{n^2}$

f) For $k=2,\dots,n+1$: $P(X+Y=k) = \sum_{i=1}^{k-1} P(X=i, Y=k-i) = (k-1)\cdot\frac{1}{n^2} = \frac{k-1}{n^2}$

For $k=n+2,\dots,2n$: $P(X+Y=k) = \sum_{i=k-n}^{n} P(X=i, Y=k-i) = [n-(k-n)+1]\cdot\frac{1}{n^2} = \frac{2n-k+1}{n^2}$

§3.2 (2) Let $x_1,\dots,x_{100}$ be the first list, $y_1,\dots,y_{100}$ the second list, and $z_1,\dots,z_{100}$ the third.

a) $\bar z = (z_1+\dots+z_{100})/100 = (x_1+y_1+\dots+x_{100}+y_{100})/100 = (x_1+\dots+x_{100})/100 + (y_1+\dots+y_{100})/100$

$= \bar x + \bar y = (0.2\times 1 + 0.8\times 2) + (0.5\times 3 + 0.5\times 5) = 1.8 + 4 = 5.8$

(Note: no assumptions were made regarding the order of ones, twos, threes and fives in the lists!)

b) Similarly, if $z_i = x_i - y_i$ for $i=1,\dots,n$, then $\bar z = \bar x - \bar y = 1.8 - 4 = -2.2$

c) & d) If $z_i = x_i y_i$ or $z_i = x_i/y_i$ for $i=1,\dots,n$, then $\bar z$ depends on the order within the lists! e.g. for $z_i = x_i y_i$,

Case 1.

x_i: $1, 1, \dots, 1, 2, 2, \dots\dots 2$

y_i: $3, 3, \dots 3, 3, \dots, 3, 5, 5, \dots, 5$

z_i: $\underbrace{3, 3, \dots, 3}_{20}, \underbrace{6, \dots, 6}_{30}, \underbrace{10, \dots, 10}_{50}$ $\Rightarrow \bar z = 7.4$

Case 2.

x_i: $2, 2, \dots\dots, 2, 1, 1, \dots, 1$

y_i: $3, 3, \dots, 3, 5, \dots, 5, 5, \dots, 5$

z_i: $\underbrace{6, 6, \dots, 6}_{50}, \underbrace{10, \dots, 10}_{30}, \underbrace{5, \dots, 5}_{20}$ $\Rightarrow \bar z = 7$