§6.2 ① a) First, for $x = 1, 2, \ldots, 6$, $P(X=x) = P(\min\{X_1, X_2\} = x)$

$$= P(X_1 = x, X_2 \geq x) + P(X_1 > x, X_2 = x) = \frac{1}{6} \cdot \frac{6-x+1}{6} + \frac{6-x}{6} \cdot \frac{1}{6}$$

$$= \frac{12-2x+1}{36} = \frac{13-2x}{36}$$

Next, $P(Y=x \mid X=x) = \dfrac{P(Y=x, X=x)}{P(X=x)} = \dfrac{P(X_1=x, X_2=x)}{P(X=x)}$

$$= \frac{\frac{1}{6} \times \frac{1}{6}}{\frac{12-2x+1}{36}} = \frac{1}{12-2x+1} = \frac{1}{13-2x}$$

and for $y = x+1, \ldots, 6$,

$$P(Y=y \mid X=x) = \frac{P(Y=y, X=x)}{P(X=x)}$$

$$= \frac{P(X_1=x, X_2=y) + P(X_1=y, X_2=x)}{P(X=x)}$$

$$= \frac{2}{13-2x}$$

Thus,

$$E(Y \mid X=x) = \sum_{y=x}^{6} y\, P(Y=y \mid X=x)$$

$$= x\, P(Y=x \mid X=x) + \sum_{y=x+1}^{6} y\, P(Y=y \mid X=x)$$

$$= \frac{x}{13-2x} + \frac{2}{13-2x} \sum_{y=x+1}^{6} y$$

$$= \frac{x}{13-2x} + \frac{2}{13-2x}\left(21 - \sum_{y=1}^{x} y\right)$$

$$= \frac{x}{13-2x} + \frac{2}{13-2x}\left(21 - \frac{x(x+1)}{2}\right)$$

$$= \frac{x + 42 - x(x+1)}{13-2x} = \frac{42 - x^2}{13-2x}$$

b) $E(X \mid Y=y)$ is calculated in a similar way!

§6.2 ⑥ a) $P(\text{all } X\text{'s} \le t \mid N=n) = P(X_1 \le t) \cdots P(X_n \le t) = t^n$.

b) $P(\text{all } X\text{'s} \le t) = \sum_{n=0}^{\infty} P(\text{all } X\text{'s} \le t \mid N=n)\, P(N=n)$

$$= \sum_{n=0}^{\infty} t^n e^{-\mu} \frac{\mu^n}{n!} = \sum_{n=0}^{\infty} e^{-\mu} \frac{(\mu t)^n}{n!} = e^{-\mu} e^{\mu t}$$

$$= e^{-\mu(1-t)}$$

c) $P(S_N = 0) = P(N=0) = e^{-\mu}$, since if $N \ge 1$, then $S_N \ge S_1 = X_1$ and $P(X_1 > 0) = 1$.

d) $E(S_N) = \sum_{n=0}^{\infty} E(S_N \mid N=n)\, P(N=n)$

$$= \sum_{n=0}^{\infty} E(S_n \mid N=n)\, P(N=n)$$

$$= \sum_{n=0}^{\infty} E(X_1 + \cdots + X_n \mid N=n)\, P(N=n)$$

$$\overset{(*)}{=} \sum_{n=0}^{\infty} E(X_1 + \cdots + X_n)\, P(N=n)$$

$$= \sum_{n=0}^{\infty} \frac{n}{2} P(N=n) = \frac{1}{2} E(N) = \frac{\mu}{2}$$

(*): assuming that $X_1, X_2, \ldots$ are independent from N.

§6.3 ② a) For $0 < x < 1$, $f_X(x) = \int_0^1 f(x,y)\,dy = \int_0^1 (2x + 2y - 4xy)\,dy$

$= \left[2xy + y^2 - 2xy^2\right]_0^1 = 2x + 1 - 2x = 1$ ($X \sim \text{uniform}(0,1)$).

For $0 < y < 1$, $f_Y(y) = \int_0^1 (2x + 2y - 4xy)\,dx = 1$ ($Y \sim \text{uniform}(0,1)$).

b) $f_Y(y \mid X = \frac{1}{4}) = \dfrac{f(\frac{1}{4}, y)}{f_X(\frac{1}{4})} = f(\frac{1}{4}, y) = \frac{1}{2} + 2y - y = y + \frac{1}{2}$

c) $E(Y \mid X = \frac{1}{4}) = \int_0^1 y\, f_Y(y \mid X = \frac{1}{4})\,dy = \int_0^1 y(y + \frac{1}{2})\,dy$

$= \int_0^1 (y^2 + \frac{1}{2}y)\,dy = \frac{1}{3} + \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{3} + \frac{1}{4} = \frac{7}{12}$.

(<u>Remark</u>: A joint density on $[0,1]^2$ of this type, with uniform marginal densities, is called a <u>copula</u>, and used to encode the dependence of two random variables with given marginal distributions.)

§6.3 ① $P(A) = \int_0^1 P(A \mid X=x)\, f_X(x)\,dx = \int_0^1 x^2 \cdot 1\,dx = \int_0^1 x^2\,dx = \frac{1}{3}$.