§4.4 ⑥ $f_\Phi(\phi) = \frac{1}{\pi}$ for $-\frac{\pi}{2} < \phi < \frac{\pi}{2}$, and $=0$ elsewhere. Thus, for $y \in \mathbb{R}$,

$$F_Y(y) = P(Y \le y) = P(\tan \Phi \le y) = P(\Phi \le \arctan y)$$
$$= \frac{1}{\pi}\left(\arctan y - \left(-\frac{\pi}{2}\right)\right) = \frac{1}{\pi}\arctan y + \frac{1}{2},$$

and

$$f_Y(y) = F_Y'(y) = \frac{1}{\pi} \cdot \frac{1}{1+y^2}.$$

Since $f_Y(-y) = \frac{1}{\pi} \cdot \frac{1}{1+(-y)^2} = \frac{1}{\pi} \cdot \frac{1}{1+y^2} = f_Y(y)$, this density is symmetric about 0. However, the integral

$$\int_{-\infty}^{\infty} y f_Y(y)\,dy = \int_{-\infty}^{\infty} \frac{y}{\pi(1+y^2)}\,dy$$

fails to converge, since both integrals $\int_{-\infty}^{0} \frac{y}{\pi(1+y^2)}\,dy$ and $\int_{0}^{\infty} \frac{y}{\pi(1+y^2)}\,dy$ diverge (e.g. by the integral test).

§4.4 (10) c) $Z \sim$ normal$(0,1)$, $Y = \frac{1}{Z}$. Range$(Y) = (-\infty,0) \cup (0,\infty)$.

(1) Let $y > 0$. Then

$$\begin{aligned}
F_Y(y) &= P(Y \le y) = P\left(\tfrac{1}{Z} \le y\right) = P\left(Z>0, \tfrac{1}{Z} \le y\right) + P(Z<0) \\
&= P\left(Z \ge \tfrac{1}{y}, Z>0\right) + P(Z<0) \\
&= P\left(Z \ge \tfrac{1}{y}\right) + \tfrac{1}{2} = \left\{1 - \Phi\left(\tfrac{1}{y}\right)\right\} + \tfrac{1}{2} \\
&= \tfrac{3}{2} - \Phi\left(\tfrac{1}{y}\right)
\end{aligned}$$

(2) Let $y < 0$. Then

$$\begin{aligned}
F_Y(y) &= P\left(\tfrac{1}{Z} \le y\right) = P\left(\tfrac{1}{Z} \le y, Z<0\right) = P(1 \ge yZ, Z<0) \\
&= P\left(\tfrac{1}{y} \le Z, Z<0\right) = \Phi(0) - \Phi\left(\tfrac{1}{y}\right) \\
&= \tfrac{1}{2} - \Phi\left(\tfrac{1}{y}\right)
\end{aligned}$$

In both cases, $f_Y(y) = F_Y'(y) = -\phi\left(\frac{1}{y}\right) \cdot -\frac{1}{y^2} = \frac{1}{y^2}\phi\left(\frac{1}{y}\right)$

$$\therefore \quad f_Y(y) = \frac{1}{y^2\sqrt{2\pi}} e^{-\frac{1}{2y^2}}, \qquad y \ne 0.$$

§5.1 ④ b) $P\left(\left|\frac{X}{Y}-1\right| \le 0.25\right) = P\left(0.75 \le \frac{X}{Y} \le 1.25\right)$

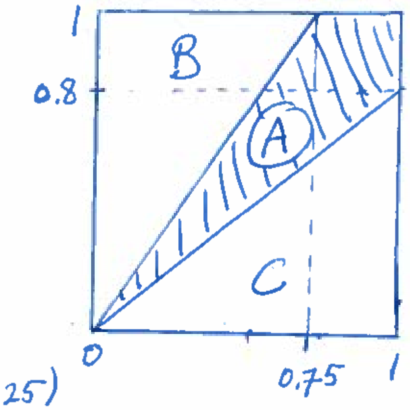
$= P(0.75Y \le X \le 1.25Y)$ (since $Y>0$)

$= P((X,Y)\in A) = \text{area}(A)$

$= 1 - \text{area}(B) - \text{area}(C)$

$= 1 - \frac{1}{2}(0.75)(1) - \frac{1}{2}(1)(0.8)$

$= 1 - 0.375 - 0.4 = 0.225$

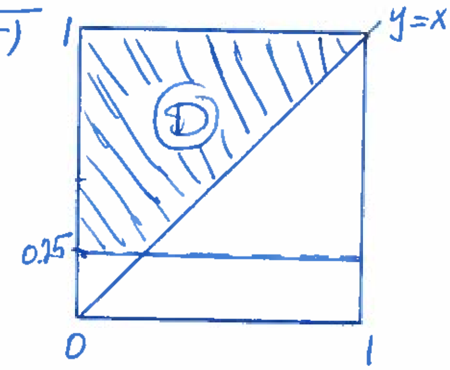


c) $P(Y \ge X \mid Y \ge 0.25) = \dfrac{P(Y\ge X,\ Y \ge 0.25)}{P(Y \ge 0.25)}$

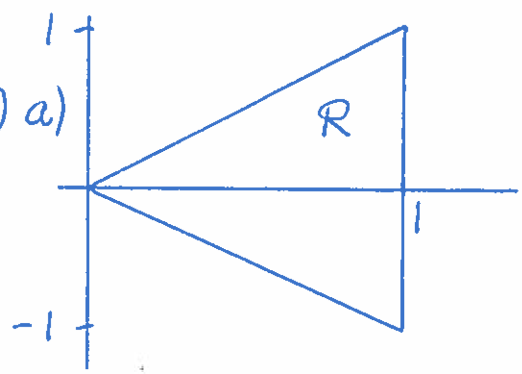
$= \dfrac{P((X,Y)\in D)}{0.75}$

$= \frac{4}{3}\left(\frac{1}{2} - \frac{1}{2}\times 0.25 \times 0.25\right)$

$= \frac{4}{3}\left(\frac{1}{2} - \frac{1}{32}\right) = \frac{4}{3}\times\frac{15}{32} = \frac{5}{8}$



§5.2 ① a)

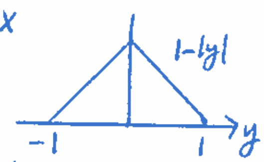


area$(R) = 1$, so (X,Y) has joint density

$$f(x,y) = \begin{cases} 1, & 0<|y|<x<1 \\ 0, & \text{elsewhere} \end{cases}$$

b) For $0<x<1$, $f_X(x) = \int_{-\infty}^{\infty} f(x,y)\,dy = \int_{-x}^{x} 1\,dy = 2x$

For $-1<y<1$, $f_Y(y) = \int_{-\infty}^{\infty} f(x,y)\,dx = \int_{|y|}^{1} 1\,dx = 1-|y|$



c) X and Y are dependent. For instance, $P(Y>\frac{1}{2}) = \frac{1}{8}$ but $P(Y>\frac{1}{2} \mid X<\frac{1}{2}) = 0$.

d) $E(X) = \int_0^1 x f_X(x)\,dx = \int_0^1 2x^2\,dx = \frac{2}{3}$

$E(Y) = 0$ by symmetry.