Hw. #13, Answers to selected problems

§4.1 ① a) 0.001 is not in the normal table. But, $\phi(z)$ is very nearly constant on $0 \le z \le 0.001$, so with good accuracy,

$$\Phi(0.001) - \Phi(0) \approx \phi(0)\cdot(0.001) = \frac{1}{\sqrt{2\pi}}\cdot(0.001) \doteq 0.000399$$

b) similar

② a) $1 = \int_{-\infty}^{\infty} f(x)\,dx = \int_1^{\infty} \frac{c}{x^4}\,dx = \int_1^{\infty} cx^{-4}\,dx = c\cdot -\frac{1}{3}x^{-3}\Big|_1^{\infty} = \frac{c}{3}$

$\therefore\ c = 3.$

b) $E(X) = \int_1^{\infty} x f(x)\,dx = \int_1^{\infty} \frac{3}{x^3}\,dx = \int_1 3x^{-3}\,dx = -\frac{3}{2}x^{-2}\Big|_1^{\infty} = \frac{3}{2}$

c) $E(X^2) = \int_1^{\infty} x^2 f(x)\,dx = \int_1^{\infty} 3x^{-2}\,dx = -3x^{-1}\Big|_1^{\infty} = 3$

$\therefore\ Var(X) = E(X^2) - [E(X)]^2 = 3 - \left(\frac{3}{2}\right)^2 = \frac{3}{4}.$

(continued on next page)

(6) a) Note that $P(X \geq 1) = 1 - P(X \leq 1) = 1 - \frac{2}{3} = \frac{1}{3} = P(X \leq 0)$.

Since the normal density is symmetric about its mean, we conclude $\mu = \frac{1}{2}$.

Then

$$\frac{2}{3} = P(X \leq 1) = \Phi\left(\frac{1-\mu}{\sigma}\right) = \Phi\left(\frac{1}{2\sigma}\right) \Rightarrow \frac{1}{2\sigma} = \Phi^{-1}\left(\frac{2}{3}\right) \approx .43$$

$$\Rightarrow \sigma = 1.163$$

b) Now it's not immediately clear where the mean is! But:

$$\frac{1}{3} = P(X \leq 0) = \Phi\left(\frac{-\mu}{\sigma}\right) \Rightarrow \Phi\left(\frac{\mu}{\sigma}\right) = \frac{2}{3} \Rightarrow \frac{\mu}{\sigma} = .43$$

$$\frac{3}{4} = P(X \leq 1) = \Phi\left(\frac{1-\mu}{\sigma}\right) \qquad \Rightarrow \frac{1-\mu}{\sigma} = .675$$

Adding the equations gives $\frac{1}{\sigma} = 1.105$, so $\sigma = 0.905$ and $\mu = (.43)\sigma = 0.389$.

(9) $E(X_i) = \frac{1}{2}$, $Var(X_i) = \frac{1}{12}$ $\Rightarrow E(S_4) = 4 \cdot \frac{1}{2} = 2$, and, since $X_1, \ldots, X_4$ are independent, $Var(S_4) = 4\, Var(X_1) = \frac{1}{3}$, $SD(S_4) = \sqrt{\frac{1}{3}} = .577$

So

$$P(S_4 \geq 3) \approx 1 - \Phi\left(\frac{3-2}{.577}\right) = 1 - \Phi(1.73) = 1 - .9582 = .0418$$

(10) a) $\mu = 9.78$, $\sigma = .0031$. Let X be a typical measurement.

$$P(9.7840 < X < 9.8000) = \Phi\left(\frac{9.8 - 9.78}{.0031}\right) - \Phi\left(\frac{9.7840 - 9.78}{.0031}\right)$$

$$= \Phi(6.45) - \Phi(1.29) = 1.000 - 0.9015 = 0.0985$$

b) Required proportion $= P(X \leq 9.7794) = \Phi\left(\frac{9.7794 - 9.78}{.0031}\right)$

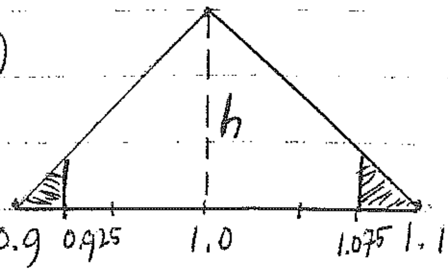
$$= \Phi(-.19) = 1 - \Phi(0.19) = 1 - .5753 = .4247.$$

c) Need w such that $P(X > w) = 0.1$, so $P(X \leq w) = 0.9$

$$\Rightarrow \Phi\left(\frac{w - 9.78}{.0031}\right) = 0.9 \Rightarrow w = 9.78 + \Phi^{-1}(0.9)(0.0031) = 9.78 + (1.28)(.0031)$$

$$= 9.783968 \approx 9.7840.$$

(13) a)



Since area of the large triangle $= 1$:
$\frac{1}{2}(1.1-0.9)\cdot h = 1 \Rightarrow (0.1)h = 1 \Rightarrow \underline{h=10}$.

By similar triangles, each shaded triangle has height $(0.025)\cdot(h/0.1) = (0.025)\cdot 100 = 2.5$, and area $\frac{1}{2}(0.025)(2.5) = .03125$.

Then, the proportion of output scrapped is the sum of the two shaded triangle areas $= 2(.03125) = 0.0625$, or $\frac{1}{16}$ or 6.25%.

b) Let X = length of a rod bought by the customer
We already know the rod wasn't scrapped, so $0.925 < X < 1.075$.
Thus we must calculate

$$p := P(0.95 < X < 1.05 \mid 0.925 < X < 1.075)$$
$$= \frac{P(0.95 < X < 1.05)}{P(0.925 < X < 1.075)} = \frac{0.75}{0.9375} = 0.8.$$

Now let n be the number ordered, and Y the number among those that satisfy the customer's requirement. Then $Y \sim \mathrm{Bin}(n, 0.8)$ so $\mu = E(Y) = 0.8n$ and $\sigma = SD(Y) = \sqrt{(0.8)(0.2)n} = 0.4\sqrt{n}$.
Then

$$P(Y \geq 100) \approx 1 - \Phi\left(\frac{100 - \frac{1}{2} - 0.8n}{0.4\sqrt{n}}\right) \geq 0.95$$

$$\Rightarrow \Phi\left(\frac{0.8n - 99.5}{0.4\sqrt{n}}\right) \geq 0.95 = \Phi(1.645)$$

$$\Rightarrow \frac{0.8n - 99.5}{0.4\sqrt{n}} \geq 1.645$$

$$\Rightarrow n \geq 134 \quad \text{(after some algebra)}.$$