Math 3000, Review for Exam 2, Solutions

1. See book
2. Let $c \in C$. Since g is surjective, there exists $b \in B$ such that $g(b) = c$. And since f is surjective, there exists further an $a \in A$ such that $f(a) = b$. Thus $(g \circ f)(a) = g(f(a)) = g(b) = c$. It follows that for every $c \in C$ there exists $a \in A$ such that $(g \circ f)(a) = c$. Hence $g \circ f$ is surjective. $\square$
3. a) False, e.g. take $x = \sqrt{2}$, $y = -\sqrt{2}$, then x and y are both irrational, but $x + y = 0$, which is rational.

   b) True. Proof: let $x \in \mathbb{Q}$ and $y \in \mathbb{R} \setminus \mathbb{Q}$. Suppose $x + y \in \mathbb{Q}$. Then $x = \frac{a}{b}$ and $x + y = \frac{c}{d}$ for certain integers a, b, c, d. But then $y = (x+y) - x = \frac{c}{d} - \frac{a}{b} = \frac{bc - ad}{bd}$, so $y \in \mathbb{Q}$, a contradiction. $\square$

4. [redacted] Complete the following table. In each cell, enter a number or write "none", as appropriate.

S	$\max S$	$\min S$	$\sup S$	$\inf S$
$(1, 5]$	5	none	5	1
$\{e, \pi, 5/2\}$	π	$5/2$	π	$5/2$
$\{(-1)^n/n : n \in \mathbb{N}\}$	$\frac{1}{2}$	-1	$\frac{1}{2}$	-1
$(0,2) = \bigcup_{n=1}^{\infty} \left(1 - \frac{1}{n}, 1 + \frac{1}{n}\right)$	none	none	2	0
$\{1\} = \bigcap_{n=1}^{\infty} \left(1 - \frac{1}{n}, 1 + \frac{1}{n}\right)$	1	1	1	1

$(e \approx 2.72)$

5. See book, p. 115
6. a) See book, p. 124

   b) See book, p. 126 below center

   c) on back

6. c) First, sup S exists by the Completeness Axiom, say $m = \sup S$. Then $m \geq s$ for all $s \in S$, so if $x \in T$ we have $\frac{1}{x} \in S$ and hence $\frac{1}{x} \leq m$, so $x \geq \frac{1}{m}$. Thus $\frac{1}{m}$ is a lower bound for T. Now let $m' \in \mathbb{R}$ such that $\frac{1}{m} < \frac{1}{m'}$ (so $m > m'$), and suppose $\frac{1}{m'}$ is a lower bound for T. Then $\frac{1}{m'} \leq \frac{1}{s}$ for all $s \in S$, and $m' \geq s$ for all $s \in S$, so m' is an upper bound for S. But $m' < m$, so this contradicts m being the <u>least</u> upper bound for S. Hence $\frac{1}{m} = \inf T$. $\square$

7. Let $\mathcal{C} = \{ (1, 5-\frac{1}{n}) \mid n \in \mathbb{N} \}$. Then $\mathcal{C}$ is an open cover for $[2,5)$ since $\bigcup_{n\in\mathbb{N}} (1, 5-\frac{1}{n}) = (1,5) \supseteq [2,5)$. But $\mathcal{C}$ does not have a finite subcover: if $\mathcal{C}' = \{ (1, 5-\frac{1}{n_1}), \dots, (1, 5-\frac{1}{n_k}) \}$ is a finite subcover, then let $n = \max\{n_1, \dots, n_k\}$. Then $\bigcup_{i=1}^{k} (1, 5-\frac{1}{n_i}) = (1, 5-\frac{1}{n})$ so $\mathcal{C}'$ does <u>not</u> cover $[2,5)$.

8. $f(-2) = (-2)^2 - 3\cdot(-2) + 7 = 4 + 6 + 7 = 17$. Then $f(x) - f(-2) = x^2 - 3x - 10$
$= (x+2)(x-5)$. Let $\delta \leq 1$ initially. Then if $|x-(-2)| < \delta$, $-3 < x < -1$ so $|x-5| < |-3-5| = 8$. Thus, take $\delta = \min\{1, \varepsilon/8\}$.

<u>Proof</u>: Given $\varepsilon > 0$, let $\delta = \min\{1, \varepsilon/8\}$. Suppose $|x-(-2)| < \delta$. Then $|x+2| < \delta$, and since $\delta \leq 1$, $|x-5| < 8$. Thus, $|f(x) - f(-2)| = |x+2||x-5| < \delta \cdot 8 \leq \varepsilon$. Hence, f is continuous at -2. $\square$

9. Let $f(x) = e^x - x^2$. Then f is continuous, $f(-1) = e^{-1} - 1 < 0$, and $f(0) = 1 - 0 = 1 > 0$. So by the intermediate value theorem, there is a number $x \in (-1, 0)$ such that $f(x) = 0$, which means $e^x = x^2$.

10. a) $f([-4,1]) = [-3, 13]$

b) $f([-1,1)) = [-3, -2]$

c) $f^{-1}([-13, 13]) = [-4, 4]$

d) $f^{-1}((0,6]) = [-3, -\sqrt{3}) \cup (\sqrt{3}, 3]$

e) $f^{-1}(\emptyset) = \emptyset$.