Math 3000 - Review for Exam 2

SHOW ALL YOUR WORK! NO WORK=NO CREDIT!!

1. Complete the following definitions:

a) A function $f : A \to B$ is *injective* iff ...
b) A function $f : A \to B$ is *surjective* iff ...
c) If $f : A \to B$, $C \subseteq A$ and $D \subseteq B$, then $f(C) = \{\dots\}$ and $f^{-1}(D) = \{\dots\}$
d) A set $S \subseteq \mathbf{R}$ is *compact* iff ...

2. Prove that if $f : A \to B$ and $g : B \to C$ are both surjective, then $g \circ f$ is surjective.

3. For each statement below, give a **proof** (if the statement is true) or a **counterexample** (if the statement is false). In each, x and y represent real numbers.

a) If x is irrational and y is irrational, then $x + y$ is irrational.
b) If x is rational and y is irrational, then $x + y$ is irrational.

4. Complete the following table. In each cell, enter a number or write "none", as appropriate.

S	$\max S$	$\min S$	$\sup S$	$\inf S$
$(1, 5]$				
$\{e, \pi, 5/2\}$				
$\{(-1)^n/n : n \in \mathbf{N}\}$				
$\bigcup_{n=1}^{\infty} \left(1 - \frac{1}{n}, 1 + \frac{1}{n}\right)$				
$\bigcap_{n=1}^{\infty} \left(1 - \frac{1}{n}, 1 + \frac{1}{n}\right)$				

5. a) Let x, y and z be real numbers. Using ONLY the field axioms, prove that

$$\text{if } x + z = y + z, \text{ then } x = y.$$

Show EVERY step, no matter how small! Indicate clearly which axiom(s) you use in each step. Avoid using more than one axiom in the same step whenever possible.

b) Using ONLY the field axioms and the result of part (a), prove that $x \cdot 0 = 0$. Show EVERY step, no matter how small! Indicate clearly which axiom(s) you use in each step. Avoid using more than one axiom in the same step whenever possible.

6. a) State as precisely as possible what it means for a number x to be the *least upper bound* of a set S.

b) State the Completeness Axiom.

c) Let S be a subset of $(0, \infty)$ and suppose S is bounded above. Define

$$T = \left\{ x \in \mathbf{R} : \frac{1}{x} \in S \right\}.$$

Prove that

$$\inf T = \frac{1}{\sup S}.$$

7. Prove that $[2, 5)$ is not compact, by identifying an open cover that does not have a finite subcover. (You may NOT use the Heine-Borel theorem!)

8. Prove that the function $f(x) = x^2 - 3x + 7$ is continuous at -2, by using an $\varepsilon - \delta$ argument.

9. Show that there is at least one real number x such that $e^x = x^2$. (Hint: use the intermediate value theorem.)

10. Let $f(x) = x^2 - 3$. Find:

a) $f([-4, 1])$

b) $f([-1, 1))$

c) $f^{-1}([-13, 13])$

d) $f^{-1}((0, 6])$

e) $f^{-1}(\emptyset)$