KEY

Name:

Math 3000 Exam 1 **October 6, 2015** **P. Allaart**

1. a) (*10 pts.*) Construct a truth table for the statement

$$[p \Rightarrow (q \wedge \sim q)] \Leftrightarrow \sim p$$

p	q	$\sim q$	$p \Rightarrow (q\wedge\sim q)$		$\Leftrightarrow$	$\sim p$
T	T	F	F	F	T	F
T	F	T	F	F	T	F
F	T	F	T	F	T	T
F	F	T	T	F	T	T

b) (*2 pts.*) Based on your truth table, is the statement in (a) a tautology?

Yes!

2. (*9 pts.*) Write the negation of the following statement such that no negation symbol ($\sim$) appears in front of quantifiers or logical connectives.

$$\forall \varepsilon > 0, \exists \delta > 0 \ni \forall x \text{ and } \forall y, [|x-y| < \delta \Rightarrow |f(x) - f(y)| < \varepsilon]$$

$$\exists \varepsilon > 0 \ni \forall \delta > 0, \exists x \text{ and } \exists y \ni [|x-y| < \delta \wedge |f(x) - f(y)| \geq \varepsilon]$$

3. (*3 pts. each*) Indicate whether each statement is true, false, or not a statement at all. (No explanations needed.)

a) $\underbrace{\text{3 is prime}}_{T}$ and $\underbrace{\text{7 is odd}}_{T}$.

True

b) 123456789 is a very big number.

Not a statement

c) $\underbrace{\pi \text{ is rational}}_{F}$ if and only if $\underbrace{\text{17 is divisible by 4}}_{F}$.

$F \Longleftrightarrow F$ True

d) If $\underbrace{\underbrace{5 < 7}_{T} \text{ implies that } \underbrace{8 > 10}_{F}}_{F}$, then $\underbrace{5/2 \text{ is an integer}}_{F}$.

$T \Rightarrow F$

$F \Rightarrow F$ True

4. (*4 pts. each*) Determine whether each statement about the real numbers is true or false. **Justify each answer carefully.**

a) $\exists x \ni [x^2 = -2 \Rightarrow x^2 = 5]$

True, e.g. take $x=0$. Then the antecedent $(x^2=-2)$ is false, so the implication is true.

b) $\exists x \ni \forall y, x + y = 0$

False: given x, we can take $y = -x+1$ and then $x+y = 1 \neq 0$.

c) $\forall x, \exists y$ and $\exists z \ni xy = z^2 + 1$

False: a counterexample is $x=0$. Then no matter how z and y are chosen, $xy = 0 < z^2+1$.

5. (*3 pts. each*) Find the following unions and/or intersections:

a) $\bigcap_{n=1}^{\infty} \left[\frac{1}{n}, 2 - \frac{1}{n}\right]$ $= \{1\}$

b) $\bigcup_{n \in \mathbb{Z}} (n, n + 1.001)$ $= \mathbb{R}$

c) $\bigcap_{x \in [1,2]} (x, x + 5)$ $= (2, 6)$

d) $\bigcup_{x \in [1,2]} (x, x + 5)$ $= (1, 7)$

6. (*3 pts. each*) For each set below, find its interior, boundary and closure:

a) $S = (1, 4)$

int $(S) =$ $(1,4)$

$\text{bd}(S) =$ $\{1,4\}$

$\text{cl}(S) =$ $[1,4]$

b) $S = \mathbb{R}$

int $(S) =$ $\mathbb{R}$

$\text{bd}(S) =$ ϕ

$\text{cl}(S) =$ $\mathbb{R}$

c) $S = \{1, 2, 3\} \cup \{x \in \mathbb{R} \mid |x - 3| < 1\}$ $= \{1,2,3\} \cup (2,4) = \{1\} \cup [2,4)$

int $(S) =$ $(2,4)$

$\text{bd}(S) =$ $\{1,2,4\}$

$\text{cl}(S) =$ $\{1\} \cup [2,4]$

d) $S = \left\{ \frac{1}{n^2} \middle| n \in \mathbb{N} \right\}$

int $(S) =$ ϕ

$\text{bd}(S) =$ $S \cup \phi$

$\text{cl}(S) =$ $S \cup \phi$

7. Recall that $A \backslash B = \{x | x \in A \text{ and } x \notin B\}$.

a) (*3 pts.*) Write the negation of the condition of membership in $A \backslash B$ (the bit behind "$x|$"). In other words, complete the following equivalence:

$x \notin A \backslash B \quad \Longleftrightarrow \quad x \notin A$ or $x \in B$

b) (*11 pts.*) Let A and B be subsets of $\mathbb{R}$. Prove that

$$A \backslash (A \backslash B) = A \cap B,$$

by proving that each side is a subset of the other. (No credit for just drawing a Venn diagram!!!)

"$\subseteq$": Let $x \in A \backslash (A \backslash B)$. Then $x \in A$ and $x \notin A \backslash B$.
So $x \notin A$ or $x \in B$. Since $x \in A$, we conclude $x \in B$.
Thus $x \in A$ and $x \in B$, in other words, $x \in A \cap B$.

"$\supseteq$": Let $x \in A \cap B$. Then $x \in A$ and $x \in B$.
Since $x \in B$, $x \notin A \backslash B$. Hence, $x \in A \backslash (A \backslash B)$.

8. a) (*3 pts.*) Write what it means, *by definition*, for x to be an interior point of a set S.

$x \in \text{int}(S)$ iff some neighborhood of x is contained in S

$(\exists\, \varepsilon > 0 \ni N(x;\varepsilon) \subseteq S)$

b) (*10 pts.*) Let S and T be subsets of $\mathbb{R}$. Prove that

$$\text{int}\,(S) \cup \text{int}\,(T) \subseteq \text{int}\,(S \cup T).$$

Let $x \in \text{int}(S) \cup \text{int}(T)$. Then $x \in \text{int}(S)$ or $x \in \text{int}(T)$.

If $x \in \text{int}(S)$, then there is $\varepsilon > 0$ such that $N(x;\varepsilon) \subseteq S$.

But then certainly $N(x;\varepsilon) \subseteq S \cup T$, since $S \subseteq S \cup T$.

So $x \in \text{int}(S \cup T)$.

Similarly, if $x \in \text{int}(T)$ then there is $\varepsilon > 0$ such that

$N(x;\varepsilon) \subseteq T \subseteq S \cup T$, so again $x \in \text{int}(S \cup T)$.

c) (*4 pts.*) Give an example to show that the statement in (b) becomes false if "$\subseteq$" is replaced with "$=$". Justify your answer!

e.g. $S = (0,1]$, $T = (1,2)$. Then

$\text{int}(S) \cup \text{int}(T) = (0,1) \cup (1,2)$, but

$\text{int}(S \cup T) = \text{int}\,(0,2) = (0,2) \neq \text{int}(S) \cup \text{int}(T)$.

9. **Extra credit!!** (*8 pts.*)
Let S and T be subsets of $\mathbb{R}$. Prove that

$$\mathrm{bd}(S \cup T) \subseteq \mathrm{bd}(S) \cup \mathrm{bd}(T).$$

Show ALL of the details!

Let $x \in \mathrm{bd}(S \cup T)$, and suppose $x \notin \mathrm{bd}(S)$.
We'll show $x \in \mathrm{bd}(T)$.

For each $\varepsilon > 0$, $N(x;\varepsilon) \cap (S \cup T) \neq \phi$ and $N(x;\varepsilon) \cap (\mathbb{R} \setminus (S \cup T)) \neq \phi$. But $\mathbb{R} \setminus (S \cup T) = (\mathbb{R} \setminus S) \cap (\mathbb{R} \setminus T)$ so for each $\varepsilon > 0$, $N(x;\varepsilon) \cap (\mathbb{R} \setminus S) \neq \phi$ and $\underbrace{N(x;\varepsilon) \cap (\mathbb{R} \setminus T) \neq \phi}_{(*)}$.

Since $x \notin \mathrm{bd}(S)$, there must exist $\varepsilon_0 > 0$ such that $N(x;\varepsilon_0) \cap S = \phi$. Now let $N(x;\varepsilon)$ be an arbitrary neighborhood of x, and consider two cases:

Case 1. If $\varepsilon \le \varepsilon_0$, then $N(x;\varepsilon) \subseteq N(x;\varepsilon_0)$ so $N(x;\varepsilon) \cap S = \phi$.
But $N(x;\varepsilon) \cap (S \cup T) \neq \phi$, so $N(x;\varepsilon) \cap T \neq \phi$.

Case 2. If $\varepsilon > \varepsilon_0$, then $N(x;\varepsilon) \supseteq N(x;\varepsilon_0)$ and $N(x;\varepsilon_0) \cap T \neq \phi$ by Case 1. Thus, $N(x;\varepsilon) \cap T \neq \phi$.

In both cases, $N(x;\varepsilon) \cap T \neq \phi$. By $(*)$, also $N(x;\varepsilon) \cap (\mathbb{R} \setminus T) \neq \phi$. Since $N(x;\varepsilon)$ was an arbitrary neighborhood of x, this shows $x \in \mathrm{bd}(T)$, as required. $\square$