Math 3000, Homework assignment #11

1. Section 3.5: 1de,2ad,4. (You may use the Heine-Borel theorem)
2. Which of the following sets are compact, which are not? Explain briefly. (You may use the Heine-Borel theorem)

   a) $(-1, 4]$

   b) $[-10^{12}, 10^{12}]$

   c) $\mathbb{Z}$

   d) $\displaystyle\bigcup_{n=1}^{\infty}\left[1-\frac{1}{n}, 1+\frac{1}{n}\right]$

   e) $\{x \in \mathbb{R} \,|\, x^2 - 2x \geq 7\}$

   f) $\left\{\frac{(-1)^n}{n^2} \mid n \in \mathbb{N}\right\} \cup \{0\}$

   g) $\emptyset$

   h) $\{1, 2, \ldots, 10\}$

3. (This problem and the next have nothing to do with compactness!) Let $f : \mathbb{R} \to \mathbb{R}$ be a function. Prove that f is continuous if and only if $f^{-1}(U)$ is open for each open set U. (Hint for "$\Leftarrow$": Let $x \in \mathbb{R}$ and $\varepsilon > 0$. Then $U = N(f(x), \varepsilon)$ is an open set, so $V = f^{-1}(U)$ is an open set that contains x. How does this give you a $\delta > 0$?)
4. Give an example to show that if $f : \mathbb{R} \to \mathbb{R}$ is continuous and $U \subseteq \mathbb{R}$ is open, then $f(U)$ is not necessarily open.
5. Let $S \subseteq \mathbb{R}$. Prove that S is compact if and only if every infinite subset of S has an accumulation point in S.

   (*Hint*: First, assume S is compact. Let T be an infinite subset of S. Then T has an accumulation point (why?), say x. But then x is an accumulation point of S (why?), and so $x \in S$ (why?).

   Conversely, assume every infinite subset of S has an accumulation point in S. Show first S is closed. Let $x \in S'$. Then for each $n \in \mathbb{N}$ there is a point $x_n \in N^*(x; 1/n) \cap S$. We may assume $x_n \neq x_m$ if $n \neq m$ (why?). Let $T = \{x_1, x_2, \ldots\}$. Then T is an infinite subset of S with exactly one accumulation point, namely x. Therefore, $x \in S$, and S is closed.

   Show next that S is bounded, by contradiction. Assume S is not bounded, say it is not bounded above. (The argument is similar if S is not bounded below.) Choose a point $x_1 \in S$. Then there is a point $x_2 \in S$ with $x_2 > x_1 + 1$ (why?). Similarly, there is $x_3 \in S$ such that $x_3 > x_2 + 1$. Continuing this way, obtain a set $T = \{x_1, x_2, \ldots\}$ of points in S such that $x_{n+1} > x_n + 1$ for all n. Explain why the existence of this set T leads to a contradiction.)
6. Turn in all of the above.