Math 3000 - Review for Final Exam **May 2017** **P. Allaart**

SHOW ALL YOUR WORK! NO WORK=NO CREDIT!!

1. Write the negation of each statement:

   a) 7 is prime and 8 is a perfect square.

   b) If $x < 2$, then $f(x) > 10$.

   c) $\forall \varepsilon > 0, \exists \delta > 0 \ni |x| < \delta \,\text{implies}\, x^2 < \varepsilon$

2. Construct a truth table for the statement $[\sim p \wedge (p \vee q)] \Rightarrow q$.

3. For each $n \in \mathbb{N}$, let $A_n = (-1/n, 1]$.

   a) $\bigcup_{n=1}^{\infty} A_n =$

   b) $\bigcap_{n=1}^{\infty} A_n =$

4. Let $f : A \to B$ and $g : B \to C$ be functions.

   a) Prove that if $g \circ f$ is injective, then f is injective.

   b) Give an example of sets A, B and C and functions $f : A \to B$ and $g : B \to C$ such that $g \circ f$ is surjective, but f is not surjective.

5. Define the symmetric difference $A\Delta B$ of subsets of a universal set U by

$$A\Delta B = (A \backslash B) \cup (B \backslash A).$$

Prove that for all subsets A, B and C of U,

$$A\Delta C \subseteq (A\Delta B) \cup (B\Delta C).$$

6. Let $\mathbb{Q}^+ = \{r \in \mathbb{Q} : r > 0\}$. Prove that $\mathbb{Q}^+$ is countable, by showing a way to enumerate (list) all of its elements.

7. Prove by induction that

$$\left(1 - \frac{1}{2^2}\right)\left(1 - \frac{1}{3^2}\right)\left(1 - \frac{1}{4^2}\right)\cdots\left(1 - \frac{1}{n^2}\right) = \frac{n+1}{2n}$$

for every $n \in \mathbb{N}$ with $n \geq 2$.

8. For each of the following sets, give the supremum, the maximum, the infimum and the minimum, or write "does not exist".

   a) $(0, 100)$

   b) $\{1/n : n \in \mathbb{N}\}$

   c) $\{|x| : -2 \leq x < 3\}$.

9. a) State the definition of a boundary point.

b) State the definition of an interior point.

c) Find the interior and the boundary of the set $S = (0,1) \cup \{r \in \mathbb{Q} : r > 1\}$. Explain your answer carefully!

10. Specify an open cover of the interval $[0,5)$ that does not have a finite subcover. Explain.

11. Consider the statement (in which x, y and z are real numbers):

$$\exists x \ni \forall y, \exists z \ni x^2 + 1 = yz \tag{1}$$

a) State the negation of statement (1).

b) Determine whether statement (1) is true or false. Explain as precisely as possible!

12. State the completeness axiom for $\mathbb{R}$.

13. Prove using an $\varepsilon - \delta$ argument: If $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ are both continuous at $a \in \mathbb{R}$, then $f + g$ is continuous at a.

14. Prove that if a nonempty set S in $\mathbb{R}$ is bounded above, then $\sup(S)$ is a boundary point of S.

15. State whether each set below is countable or uncountable.

a) $\mathbb{Z}$

b) $[1,2)$

c) $\{r + \sqrt{2} \,|\, r \in \mathbb{Q}\}$

d) $\{1,2,3\}$

16. Prove using ONLY the definition of a compact set (NOT the Heine-Borel theorem!) that if S and T are compact sets, then $S \cup T$ is compact.