Math 3000, Review for final exam, Solutions.

1. a) 7 is not prime or 8 is not a perfect square
   b) $x < 2$ and $f(x) \leq 10$
   c) $\exists\ \varepsilon > 0 \ni \forall\ \delta > 0, \exists\ x \ni [\,|x| < \delta \wedge x^2 \geq \varepsilon\,]$
2. Straightforward
3. a) Note that $A_1 \supseteq A_2 \supseteq \dots \supseteq A_n \supseteq A_{n+1} \supseteq \cdots$, so $\bigcup_{n=1}^{\infty} A_n = A_1 = (-1,1]$
   b) Since $\lim_{n\to\infty} \frac{1}{n} = 0$, and $0 \in A_n$ for each n, $\bigcap_{n=1}^{\infty} A_n = [0,1]$
4. a) Suppose $f(x) = f(y)$. Then $g(f(x)) = g(f(y))$; that is, $g \circ f(x) = g \circ f(y)$. Since $g \circ f$ is injective, $x = y$. Therefore, f is injective.
   b) For example, let $A = B = \mathbb{R}$, $C = \{0\}$, $f(x) = x^2$ and $g(x) = 0$.
5. Let $x \in A \Delta C$. Then $x \in A \setminus C$ or $x \in C \setminus A$. Suppose $x \in A \setminus C$. Then $x \in A$ but $x \notin C$. If $x \notin B$ then $x \in A \setminus B$ so $x \in A \Delta B$. On the other hand, if $x \in B$ then $x \in B \setminus C$ so $x \in B \Delta C$. In either case, $x \in (A \Delta B) \cup (B \Delta C)$. The same conclusion (by symmetry) follows if $x \in C \setminus A$. Thus, $A \Delta C \subseteq (A \Delta B) \cup (B \Delta C)$.
6. Done in class.
7. $P(2)$ says that $1 - \frac{1}{2^2} = \frac{2+1}{2 \cdot 2}$, which is true since $1 - \frac{1}{2^2} = \frac{3}{4} = \frac{2+1}{2\cdot 2}$.
   Let $n \in \mathbb{N}$, $n \geq 2$ be arbitrary and assume $P(n)$ is true. Then

$$\left(1-\frac{1}{2^2}\right)\left(1-\frac{1}{3^2}\right)\left(1-\frac{1}{4^2}\right)\cdots\left(1-\frac{1}{(n+1)^2}\right) = \left(1-\frac{1}{2^2}\right)\left(1-\frac{1}{3^2}\right)\left(1-\frac{1}{4^2}\right)\cdots\left(1-\frac{1}{n^2}\right)\left(1-\frac{1}{(n+1)^2}\right)$$

$$= \frac{n+1}{2n}\cdot\left(1-\frac{1}{(n+1)^2}\right) \qquad \text{(by the induction hypothesis)}$$

$$= \frac{n+1}{2n}\left(\frac{(n+1)^2-1}{(n+1)^2}\right) = \frac{n+1}{2n}\cdot\frac{n^2+2n}{(n+1)^2} = \frac{n+1}{2n}\cdot\frac{n(n+2)}{(n+1)^2} = \frac{n+2}{2(n+1)}.$$

   Thus, $P(n+1)$ is true. By the principle of induction, $P(n)$ is true for every $n \geq 2$.
8. a) $\sup S = 100$, $\inf S = 0$, $\max S$ d.n.e., $\min S$ d.n.e.
   b) $\sup S = \max S = 1$, $\inf S = 0$, $\min S$ d.n.e.
   c) $S = [0,3)$ $\therefore$ $\sup S = 3$, $\inf S = \min S = 0$, $\max S$ d.n.e.
9. a) & b): See book, § 13.4
   c) $\operatorname{int} S = (0,1)$ since $\{r \in \mathbb{Q} : r > 1\}$ has no interior points.
   $\operatorname{bd} S = \{0\} \cup [1,\infty)$ since for every $x \in [1,\infty)$, each neighborhood of x contains rational as well as irrational numbers.
10. For instance, let $A_n = (-1, 5 - \frac{1}{n})$, for $n \in \mathbb{N}$, and $\mathcal{F} = \{A_n : n \in \mathbb{N}\}$.

11. a) $\forall x\ \exists y \ni \forall z,\ x^2+1 \neq yz$

b) The statement is false, since its negation is true: no matter what x is, take $y=0$, then for every z, $x^2+1>0$ but $yz=0$, so $x^2+1\neq yz$.

12. See book, p. 126.

13. Let $\varepsilon>0$ be given. Since f is continuous at a, there exists $\delta_1>0$ s.t. if $|x-a|<\delta_1$, then $|f(x)-f(a)|<\frac{\varepsilon}{2}$. Similarly, since g is continuous at a, there exists $\delta_2>0$ s.t. if $|x-a|<\delta_2$, then $|g(x)-g(a)|<\frac{\varepsilon}{2}$. Take $\delta=\min\{\delta_1,\delta_2\}$. Assume $|x-a|<\delta$. Then $|x-a|<\delta_1$ and so $|f(x)-f(a)|<\frac{\varepsilon}{2}$. Likewise, $|x-a|<\delta_2$, and so $|g(x)-g(a)|<\frac{\varepsilon}{2}$. But then

$$\begin{aligned}|(f+g)(x)-(f+g)(a)| &= |f(x)+g(x)-(f(a)+g(a))|\\ &= |f(x)-f(a)+g(x)-g(a)|\\ &\le |f(x)-f(a)|+|g(x)-g(a)|\\ &< \frac{\varepsilon}{2}+\frac{\varepsilon}{2}=\varepsilon.\end{aligned}$$

Therefore, $f+g$ is continuous at a. $\square$

14. First, $m=\sup(S)$ exists by the Completeness Axiom. We must show that $\forall\varepsilon>0$, $N(m;\varepsilon)\cap S\neq\phi$ and $N(m;\varepsilon)\cap(\mathbb{R}\setminus S)\neq\phi$. Let $\varepsilon>0$ be given. Since m is an upper bound for S, $m+\frac{\varepsilon}{2}\notin S$ so $m+\frac{\varepsilon}{2}\in N(m;\varepsilon)\cap(\mathbb{R}\setminus S)$, and $N(m;\varepsilon)\cap(\mathbb{R}\setminus S)\neq\phi$. On the other hand, $m-\varepsilon$ is not an upper bound for S, so there is a point $x\in(m-\varepsilon,m]\cap S$ and hence $N(m;\varepsilon)\cap S\neq\phi$. Thus, $m\in\operatorname{bd}(S)$. $\square$

15. a) countable b) uncountable c) countable* d) countable

*: since the function $f:\mathbb{Q}\to\{r+\sqrt{2}\mid r\in\mathbb{Q}\}$, $f(r)=r+\sqrt{2}$ is a bijection.

16. Let $\mathcal{G}$ be an open cover for $S\cup T$. Then $\mathcal{G}$ is in particular an open cover for S, and since S is compact, $\mathcal{G}$ has a finite subcover $\mathcal{G}_1'$ for S. In the same way, $\mathcal{G}$ has a finite subcover $\mathcal{G}_2'$ for T. But then $\mathcal{G}_1'\cup\mathcal{G}_2'$ is a finite subcover of $\mathcal{G}$ for $S\cup T$. Hence, $S\cup T$ is compact.