Math 2730 Exam 1, Solutions (9/25/2014)

1. The midpoint of PQ and center of the sphere is $M(1,2,3)$. The radius is $r = |PM| = \sqrt{(1-1)^2+(2-1)^2+(3-1)^2} = \sqrt{5}$. Thus, the equation for the sphere is:

$$(x-1)^2+(y-2)^2+(z-3)^2 = 5.$$

2. $\int \left(e^{2t}\vec{i} + (t+t^{-1})\vec{j} + \frac{1}{1+t^2}\vec{k}\right)dt = \frac{1}{2}e^{2t}\vec{i} + \left(\frac{t^2}{2}+\ln|t|\right)\vec{j} + (\arctan t)\vec{k} + \vec{C}$

3. a) $\vec{AB} = \langle 0,2,4\rangle$, $\vec{AC} = \langle 2,-1,1\rangle \Rightarrow \vec{AB}\times\vec{AC} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 0 & 2 & 4 \\ 2 & -1 & 1 \end{vmatrix}$

$= \begin{vmatrix} 2 & 4 \\ -1 & 1 \end{vmatrix}\vec{i} - \begin{vmatrix} 0 & 4 \\ 2 & 1 \end{vmatrix}\vec{j} + \begin{vmatrix} 0 & 2 \\ 2 & -1 \end{vmatrix}\vec{k} = 6\vec{i}+8\vec{j}-4\vec{k}$, and then

area of $\triangle ABC = \frac{1}{2}|\vec{AB}\times\vec{AC}| = \frac{1}{2}\sqrt{6^2+8^2+4^2} = \frac{1}{2}\sqrt{116} = \boxed{\sqrt{29}}$

b) $\vec{AB}\cdot\vec{AC} = 0\cdot 2 + 2\cdot(-1) + 4\cdot 1 = 2$, $|\vec{AB}| = \sqrt{20}$, $|\vec{AC}| = \sqrt{6}$

$$\Rightarrow \cos\angle A = \frac{\vec{AB}\cdot\vec{AC}}{|\vec{AB}||\vec{AC}|} = \frac{2}{\sqrt{20}\sqrt{6}} = \frac{1}{\sqrt{30}}$$

$$\Rightarrow \angle A = \arccos\left(\frac{1}{\sqrt{30}}\right) \approx 79^\circ.$$

4. a) true b) false (e.g. take $\vec{u} = \vec{v} = \vec{i}$, $\vec{w} = \vec{j}$)
c) false (correct is: $\vec{u}\cdot\vec{u} = |\vec{u}|^2$) d) true (scalar triple product)

5. a) $\vec{PQ} = \langle 1,1,2\rangle$, $\vec{PR} = \langle -1,0,-1\rangle \Rightarrow \vec{PQ}\cdot\vec{PR} = -3$, $|\vec{PR}| = \sqrt{2}$

$$\therefore \operatorname{proj}_{\vec{PR}}\vec{PQ} = \left(\frac{\vec{PQ}\cdot\vec{PR}}{|\vec{PR}|^2}\right)\vec{PR} = \frac{-3}{2}\langle -1,0,-1\rangle = \left\langle \frac{3}{2}, 0, \frac{3}{2}\right\rangle.$$

b) any vector $\vec{u}$ with $\vec{u}\cdot\vec{PR} = 0$ and $|\vec{u}| = 1$, e.g. $\vec{u} = \langle 0,1,0\rangle$ or $\vec{u} = \left\langle \frac{1}{\sqrt{2}}, 0, -\frac{1}{\sqrt{2}}\right\rangle$.

6. a) $x(t) = (v_0\cos\alpha)t$, $y(t) = (v_0\sin\alpha)t - \frac{1}{2}gt^2$ where $g = 32$ ft/sec^2.
To find the range, set $y(t) = 0$: this gives ($t=0$ or) $t = \frac{2v_0\sin\alpha}{g}$. Then

$$\text{Range} = x(t) = v_0\cos\alpha\cdot\frac{2v_0\sin\alpha}{g} = \frac{v_0^2\sin 2\alpha}{g} = \frac{90^2\cdot\sin 90^\circ}{32} = \frac{8100}{32} = 253.125 \text{ ft}$$

Thus, the boulder will <u>not</u> hit the castle and fall 46.875 ft short.

b) $x(t) = (v_0\cos\alpha)t$, $y(t) = (v_0\sin\alpha)t - \frac{1}{2}gt^2$, so $y = (\tan\alpha)x - \frac{gx^2}{2v_0^2\cos^2\alpha}$.
Since $\alpha = 45^\circ$, $\tan\alpha = 1$ and $\cos\alpha = \sqrt{2}/2$, so $\cos^2\alpha = \frac{1}{2}$. Here $g = 32$ ft/sec^2. Thus
$y = x - \frac{32x^2}{2v_0^2\cdot\frac{1}{2}} = x - \frac{32x^2}{v_0^2}$. Now when $x = 300$ we want $y = 30$, so $30 = 300 - \frac{32(300^2)}{v_0^2}$
$\Rightarrow v_0^2 = \frac{32\cdot 300^2}{270} \Rightarrow v_0 = 300\sqrt{\frac{32}{270}} = \boxed{103.28 \text{ ft/sec}}$.

7. a) $|\vec{r}(t)|^2 = (\sin t + \sqrt{3}\cos t)^2 + (\sqrt{3}\sin t - \cos t)^2$

$$= (\sin^2 t + 2\sqrt{3}\sin t\cos t + 3\cos^2 t) + (3\sin^2 t - 2\sqrt{3}\sin t\cos t + \cos^2 t)$$

$$= 4\sin^2 t + 4\cos^2 t \qquad \text{(middle terms cancel!)}$$

$$= 4$$

$\therefore$ $|\vec{r}(t)| = 2$ for all t, so particle moves on a circle with radius 2.

b) $\vec{v}(t) = \dfrac{d\vec{r}}{dt} = (\cos t - \sqrt{3}\sin t)\vec{i} + (\sqrt{3}\cos t + \sin t)\vec{j}$

$$\therefore \vec{r}(t)\cdot\vec{v}(t) = (\sin t + \sqrt{3}\cos t)(\cos t - \sqrt{3}\sin t) + (\sqrt{3}\cos t + \sin t)(\sqrt{3}\sin t - \cos t)$$

$$= \sin t\cos t - \sqrt{3}\sin^2 t + \sqrt{3}\cos^2 t - 3\cos t\sin t$$
$$+ 3\cos t\sin t - \sqrt{3}\cos^2 t + \sqrt{3}\sin^2 t - \sin t\cos t$$

$$= 0 \qquad \text{(everything cancels)}$$

$\therefore$ $\vec{r}(t) \perp \vec{v}(t)$.

8. a) No. For instance, take $\vec{A} = \langle 1,1\rangle$, $\vec{B}_1 = \langle 1,0\rangle$, $\vec{B}_2 = \langle 0,1\rangle$. Then $\vec{A}\cdot\vec{B}_1 = \vec{A}\cdot\vec{B}_2 = 1$, and $\vec{A} \neq \vec{0}$, but $\vec{B}_1 \neq \vec{B}_2$.

b) No. Easiest counterexample is to take $\vec{A}$, $\vec{B}_1$ and $\vec{B}_2$ all parallel and nonzero, e.g. $\vec{A} = \langle 1,0,0\rangle$, $\vec{B}_1 = \langle 2,0,0\rangle$, $\vec{B}_2 = \langle 3,0,0\rangle$. Then $\vec{A}\times\vec{B}_1 = \vec{A}\times\vec{B}_2 = \vec{0}$, $\vec{A} \neq \vec{0}$, but $\vec{B}_1 \neq \vec{B}_2$. For an example with nonparallel vectors, let $\vec{A} = \vec{i}$, $\vec{B}_1 = \vec{j}$, and $\vec{B}_2 = \vec{i} + \vec{j}$. Then $\vec{A}\times\vec{B}_1 = \vec{A}\times\vec{B}_2 = \vec{k}$ and $\vec{A} \neq \vec{0}$, but $\vec{B}_1 \neq \vec{B}_2$.