Math 2700 - Review for Final Exam (April 2013)

SHOW ALL YOUR WORK! NO WORK=NO CREDIT!!

1. Find the value of h such that the columns of

$$A = \begin{bmatrix} -1 & 1 & 5 \\ 0 & 3 & h \\ 2 & 4 & -6 \end{bmatrix}$$

are linearly *dependent*.

2. Find the general solution of the following system of equations in parametric vector form:

$$\begin{aligned} x_1 + 2x_2 - 3x_3 + x_4 &= 1 \\ -x_1 - x_2 + 4x_3 - x_4 &= 6 \\ -2x_1 - 4x_2 + 7x_3 - x_4 &= 1 \end{aligned}$$

3. Let T be a linear transformation from $\mathbb{R}^7$ into $\mathbb{R}^5$, with standard matrix A.

a) How many *columns* does A have?

b) Could the rank of A be 6? Why/why not?

c) Suppose the rank of A is 5. Explain why this means that T is onto.

4. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation that reflects a point through the line $x_2 = x_1$.

a) Find a nonzero vector $\mathbf{x}$ such that $T(\mathbf{x}) = \mathbf{x}$.

b) Find a nonzero vector $\mathbf{x}$ such that $T(\mathbf{x}) = -\mathbf{x}$.

c) What are the eigenvalues of A, the standard matrix of T? (It is NOT necessary to compute A!)

5. Let

$$A = \begin{bmatrix} 2 & -3 & 6 & 2 & 5 \\ -2 & 3 & -3 & -3 & -4 \\ 4 & -6 & 9 & 5 & 9 \\ -2 & 3 & 3 & -4 & 1 \end{bmatrix}, \qquad B = \begin{bmatrix} 2 & -3 & 6 & 2 & 5 \\ 0 & 0 & 3 & -1 & 1 \\ 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Assume that A and B are row equivalent. (They are.)

a) Give $\operatorname{rank} A$ and $\dim \operatorname{Nul} A$.

b) Find bases for $\operatorname{Col} A$ and $\operatorname{Nul} A$.

6. Calculate the determinant. You may use any method, but make sure that each step can be clearly understood!

$$\begin{vmatrix} 0 & 1 & 2 & 3 \\ 1 & 3 & 3 & 4 \\ 2 & 4 & 7 & 11 \\ 3 & 3 & 3 & 3 \end{vmatrix} =$$

7. Let $T : \mathbb{R}^3 \to \mathbb{P}_1$ be the mapping

$$T(x, y, z) = (x + y + z) + (x - y - z)t.$$

a) Show that T is a linear transformation.

b) Find a basis for the kernel of T.

8. Let $A = \begin{bmatrix} 2 & 1 \\ 7 & -4 \end{bmatrix}$. Find the eigenvalues of A, and find at least one eigenvector for each eigenvalue.

9. Suppose A, B and X are matrices such that A, X, and $I + AX$ are invertible, and suppose that

$$(I + AX)^{-1} = X^{-1}B. \tag{1}$$

a) Explain why B is invertible.

b) Solve the equation (1) for X. If you need the inverse of a matrix, explain why that matrix is invertible.

10. Diagonalize the matrix

$$A = \begin{bmatrix} -1 & 3 & 3 \\ 6 & 2 & -3 \\ -12 & 6 & 11 \end{bmatrix},$$

if possible, given that the eigenvalues of A are 2 and 5.

11. Let $\mathbb{P}_3$ denote the vector space of all polynomials with real coefficients of degree at most 3. Let H be the subset of $\mathbb{P}_3$ of odd polynomials, that is, $\mathbf{p} \in H$ if and only if $\mathbf{p}(-t) = -\mathbf{p}(t)$ for every t in $\mathbb{R}$.

Show that H is a subspace of $\mathbb{P}_3$ and give a basis for H.

12. **Extra credit!!**

Prove that for every square matrix A, A^T and A have the same eigenvalues. (*Hint*: show that A^T has the same characteristic polynomial as A.)